MAR 1 1 1982

105.A-11 Y3.N88:25/2354

NUREG/CR-2354
ANL-81-69

NUREG/CR-2354
ANL-81-69

COMPLETED
58

ORIGINAL

LOCAL VOLUME-AVERAGED TRANSPORT EQUATIONS FOR MULTIPHASE FLOW IN REGIONS CONTAINING DISTRIBUTED SOLID STRUCTURES

by

W. T. Sha, B. T. Chao,
and S. L. Soo

ARGONNE NATIONAL LABORATORY, ARGONNE, ILLINOIS

Prepared for the Office of Nuclear Regulatory Research
U. S. NUCLEAR REGULATORY COMMISSION
under Interagency Agreement DOE 40-550-75

I

The facilities of Argonne National Laboratory are owned by the United States Government. Under the terms of a contract (W-31-109-Eng-38) among the U. S. Department of Energy, Argonne Universities Association and The University of Chicago, the University employs the staff and operates the Laboratory in accordance with policies and programs formulated, approved and reviewed by the Association.

MEMBERS OF ARGONNE UNIVERSITIES ASSOCIATION

The University of Arizona
Carnegie-Mellon University
Case Western Reserve University
The University of Chicago
University of Cincinnati
Illinois Institute of Technology
University of Illinois
Indiana University
The University of Iowa
Iowa State University
The University of Kansas
Kansas State University
Loyola University of Chicago
Marquette University
The University of Michigan
Michigan State University
University of Minnesota
University of Missouri
Northwestern University
University of Notre Dame
The Ohio State University
Ohio University
The Pennsylvania State University
Purdue University
Saint Louis University
Southern Illinois University
The University of Texas at Austin
Washington University
Wayne State University
The University of Wisconsin-Madison

NOTICE

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, or any of their employees, makes any warranty, expressed or implied, or assumes any legal liability or responsibility for any third party's use, or the results of such use, of any information, apparatus, product or process disclosed in this report, or represents that its use by such third party would not infringe privately owned rights.

Available from

GPO Sales Program
Division of Technical Information and Document Control
U. S. Nuclear Regulatory Commission
Washington, D.C. 20555

and

National Technical Information Service
Springfield, Virginia 22161

NUREG/CR-2354
ANL-81-69
(Distribution
Code: R7)

ARGONNE NATIONAL LABORATORY
9700 South Cass Avenue
Argonne, Illinois 60439

LOCAL VOLUME-AVERAGED TRANSPORT EQUATIONS FOR MULTIPHASE FLOW IN REGIONS CONTAINING DISTRIBUTED SOLID STRUCTURES

by

W. T. Sha, B. T. Chao,*
and S. L. Soo*

Components Technology Division

December 1981

Prepared for

Office of Nuclear Regulatory Research
U. S. Nuclear Regulatory Commission
Washington, D. C. 20555
Under Interagency Agreement DOE 40-550-75

NRC FIN No. A2045

*Consultants; Department of Mechanical Engineering,
University of Illinois, Urbana, 61801

LOCAL VOLUME-AVERAGED TRANSPORT EQUATIONS FOR MULTIPHASE FLOW IN REGIONS CONTAINING DISTRIBUTED SOLID STRUCTURES

by

W. T. Sha, B. T. Chao, and S. L. Soo

ABSTRACT

Local volume averaged conservation equations for multiphase flow in regions containing a distributed fixed solid structure with heat transfer are treated. In addition to the usual concept of volume porosity and distributed resistance, the anisotropic nature of the porous media is accounted for by formulating the equations in terms of velocities of phases associated with directional surface permeability. The resulting basic equations are in the form of differential integral transport equations with probability integrals based on the configurations and velocities of the interface. Limiting cases of pure stratified flow and highly dispersed mixture flow are noted. Validity and basis of the governing equations used in the COMMIX-2 code at its present stage of development are demonstrated.

FIN # A2045

TITLE
3-D Time-dependent Code Development

TABLE OF CONTENTS

		PAGE
NOMENCLATURE		vi
EXECUTIVE SUMMARY		1
1.	INTRODUCTION	2
	1.1 Porosity and Surface Permeability	2
	1.2 An Illustration of the Significance of Phase Configuration	4
2.	AVERAGING RELATIONS	6
	2.1 Local Volume Average, Intrinsic Average, and Area Average	6
	2.2 Local Averaging Theorems	12
3.	DYNAMIC EQUATIONS OF A PHASE, INTERFACE, AND AVERAGES	15
	3.1 Dynamics of a Phase and an Interface	15
	3.2 Volume-Averaged Dynamic Equations of a Phase and Its Interfaces	16
	3.3 Volume-Averaged Energy Equation in Terms of Internal Energy	22
	3.4 Volume-Averaged Energy Equation in Terms of Enthalpy	24
4.	INFLUENCE OF FLOW CONFIGURATIONS	27
	4.1 Pure Stratified Flow	27
	4.2 Highly Dispersed Flow	29
	4.3 Other Limiting Conditions	30
5.	FORMULATION FOR COMMIX-2	32
6.	CONCLUDING REMARKS	35
APPENDIX	MIXTURE RELATIONS	36
1	Volume Averaging as Applied to the Mixture of Phases	36
2	Energy Equation in Terms of Internal Energy	39
3	Energy Equation in Terms of Enthalpy	41
4	Pure Stratified Flow	43
5	Highly Dispersed Flow	43
ACKNOWLEDGMENT		44
REFERENCES		45

NOMENCLATURE

NOMENCLATURE

A	Area as defined
b	Dimension as defined
c_p	Specific heat at constant pressure
C_D	Drag coefficient
D_h	Hydraulic diameter
E	Total energy, $u + \frac{1}{2}\underline{U} \cdot \underline{U}$
$\underline{f}$	Field force per unit mass or a function
$\underline{g}$	Gravitational acceleration
h	Enthalpy
H	Curvature of surface
I	Indicator function
$\underline{I}$	Unitary tensor
J_E	Heat source
$\underline{J}$	Heat flux or general flux
K	Friction function
Nu	Nusselt number for heat transfer
$\underline{n}$	Unit outward normal vector
P	Pressure
$\underline{p}$	Position vector
$\dot{Q}$	Heat-transfer rate
$\underline{R}$	Flow resistance of solid
R	Heat-transfer function
t	Time
T	Temperature
u	Internal energy
$\underline{U}$	Velocity

NOMENCLATURE (Contd.)

v	Local averaging volume
V	Viscous coupling
x,y,z	Cartesian coordinates
Γ	Mass generation rate per unit volume of fluid mixture
α	Volume fraction
γ_v	Volume porosity
γ_{Aj}	Surface permeability of an area normal to the j direction
κ	Thermal conductivity or eddy conductivity
ρ	Density
$\bar{\rho}$	Density of material
σ	Surface tension
$\underline{\tau}$	Viscous stress tensor
ψ	An intensive property
Φ	Dissipation rate per unit volume

Subscript

A	Area
e	Entrance or exit
f	Fluid phase
kf	Interface of phases k and f
wk	Interface of phase k and wall
k	A phase or bubble phase
i,j	Components
ji	Tensorial components, directed toward i in the j plane
m	Mixture, or overall fluid phase
mk	Pertaining to phase k in the mixture

NOMENCLATURE (Contd.)

P	Pressure
w	Wall or solid surface
_	Vector
=	Second-order tensor
x,y,z	Components
τ	Shear stress

Superscript

—	Material Property
3	Volume average based on total volume
3m	Volume average based on volume of fluid mixture
3i	Intrinsic average based on volume of a phase
2	Area average
2m	Area average based on area occupied by fluid mixture
2i	Area average based on area occupied by a phase

Operator

∇	Gradient
$\nabla \cdot$	Divergence
$\langle\ \rangle$	Average

EXECUTIVE SUMMARY

In general, multiphase fluid flow and heat transfer in reactor components are complex. The complication arises from the fact that the flow region of interest often contains irregularly shaped structures. The cost of detailed thermal-hydraulic analysis with explicit treatment of these structures is often prohibitive, if not impossible. An alternative is to capture the essential features of the system and to express the flow and temperature field in terms of locally global quantities while sacrificing some of the details. Oftentimes, this is all that is required in most engineering applications. The present work is an attempt to achieve this goal by applying the local volume-averaging technique. In the course of derivations, fundamental averaging theorems and their basic premises have been reconsidered.

Attention is focused on multiphase flows in a region containing dispersed but fixed heat-generating or heat-absorbing solids. Local volume-averaged governing equations, i.e., conservation of mass, momentum, and energy, and interface relations for such a system are derived. The need for introducing volume porosity and surface permeability arises naturally in the averaging process. The concept of surface permeability provides an artifice for modeling anisotropic structural resistance to flow, especially baffles and tube sheets. This surface permeability is based on the free-flow area of the enclosing surface of the local averaging volume.

The resultant set of conservation equations consists of differential-integral equations because of the existence of interface transfer integrals associated with probable interface configurations. In the limiting cases of pure stratified flow and highy dispersed flow, they are reduced to differential equations. Other interface configurations of flow regimes exist between these two limiting cases.

The approximations of the governing equations used in COMMIX-2 at the present stage of development have been delineated. Future research in improving these approximations is suggested.

1. INTRODUCTION

Present study extends the case of flow of a single-phase fluid through a porous medium (Sha and Chao, 1981) to that of a multiphase fluid mixture. The porous media representation is an approximate but useful means of large-scale numerical computation for thermal hydraulics in nuclear reactors. The multiphase fluid mixture may consist of a liquid with bubbles of various sizes, or a vapor with droplets of various sizes. The porous medium may consist of dispersed, stationary, heat-generating or -absorbing solid structures. Derivation of local volume-averaged conservation equations are based on the averaging theorems of Slattery (1967) and Whitaker (1967, 1969). The averaging process leads to a set of general differential-integral equations of multiphase flow. Relations to different flow regimes of annular, slug, froth, droplet, and bubble flow exist between these two limiting conditions.

The description of the porous media in terms of volume porosity and surface permeability makes possible the analysis of thermal hydraulics in complex nuclear reactor components. Such an approach requires some approximations, but the approximations involved are generally compatible with the finite difference numerical computation scheme using realistic computational grid size.

This report presents a rigorous derivation of the governing differential-integral conservation equations for multiphase flow and the associated interfacial conditions. The difficulty of solving these equations for the general multiphase flow is noted. Simplifications are thus introduced that lead to the equations of the COMMIX-2 code at the present stage of development. The assumptions underlying such simplifications are explained.

1.1 Porosity and Surface Permeability

Nuclear reactor components can be represented by a porous solid structure immersed in the fluid phase. The relation between volume porosity and mean flow area through porous media is seen via a simple example of a medium formed by solid cubes of sides b arranged with spacings between centers of $1/b = N_x$ cubes per meter in the x direction, and $1/(2b) = N_y$, $1/(3b) = N_z$ in the y and z directions, forming a porous medium in the form of a cube of 1 meter per side. The volume porosity γ_v is thus

$$\gamma_v = 1 - N_x N_y N_z b^3 = 5/6 \tag{1.1}$$

Properties such as density, enthalpy, and static pressure of a fluid should obviously be based on volume porosity γ_v. The mass flow in any given direction may be related to a mean interstitial velocity based on the mean flow area normal to this velocity.

The average fractional flow areas in each direction are given by

$$A_{x(av)}\Big/A = \frac{1}{3b}\ \frac{(3b - b)b + 3b(b)}{2b} = 5/6\ ,$$

$$A_{y(av)}\Big/A = \frac{1}{2b}\ \frac{(2b - b)b + (3b - b)2b}{3b} = 5/6 ,$$

and

$$A_{z(av)}\Big/A = \frac{1}{2b(3b)}\left[(2b)3b - b^2\right] = 5/6 . \tag{1.2}$$

Thus, even for an anisotropic porous medium, the mean fraction flow areas in each direction are identical and equal to γ_v. However, it is more convenient to base the computations on the free-flow area and the corresponding interstitial velocity. For the above example, the free-flow areas in each direction, referred to as surface permeability γ_{Aj} for an area whose normal is in the j direction, have the following values:

$$\left.\begin{aligned} \gamma_{Ax} &= 1 - N_y N_z b^2 = 5/6 , \\ \gamma_{Ay} &= 1 - N_z N_x b^2 = 2/3 , \\ \text{and} \\ \gamma_{Az} &= 1 - N_x N_y b^2 = 1/2 . \end{aligned}\right\} \tag{1.3}$$

The surface permeabilities so defined are vector coefficients (not vectors), the meaning of which will soon be clarified. Use of γ_{Aj} also facilitates modeling of anisotropic complex systems including baffles.

A strictly defined idealized system on which our formulation is based consists of a distributed volume porosity γ_v with distributed surface permeability components γ_{Aj}, for j = x, y, z in Cartesian coordinates such as illustrated in the above example. In such a system, the distributed velocities and viscous stresses are based on γ_{Aj}. The present formulation is not based on the mean interstitial velocities and the mean flow areas or segmental average (Sha and Chao, 1980). The meaning of the surface permeabilities as vector coefficients γ_{Aj} is, for any flow and flux property $\underline{\psi}$, a vector (Sha, 1978; Soo and Sha, 1978),

$$\nabla \cdot \gamma_A \underline{\psi} = \frac{\partial}{\partial x}\gamma_{Ax}\,\psi_x + \frac{\partial}{\partial y}\gamma_{Ay}\,\psi_y + \frac{\partial}{\partial z}\gamma_{Az}\,\psi_z \tag{1.4}$$

and for a second-order tensor, $\underline{\psi} = \psi_{ji}$,

$$\nabla \cdot \gamma_A \underline{\psi} = \underline{i}\left(\frac{\partial}{\partial x}\gamma_{Ax}\,\psi_{xx} + \frac{\partial}{\partial y}\gamma_{Ay}\,\psi_{yx} + \frac{\partial}{\partial z}\gamma_{Az}\,\psi_{zx}\right)$$

$$+ \underline{j}\left(\frac{\partial}{\partial x}\gamma_{Ax}\,\psi_{xy} + \frac{\partial}{\partial y}\gamma_{Ay}\,\psi_{yy} + \frac{\partial}{\partial z}\gamma_{Az}\,\psi_{zy}\right)$$

$$+ \frac{k}{} \left(\frac{\partial}{\partial x} \gamma_{Ax} \psi_{xz} + \frac{\partial}{\partial y} \gamma_{Ay} \psi_{yz} + \frac{\partial}{\partial z} \gamma_{Az} \psi_{zz} \right). \qquad (1.5)$$

These relations in Cartesian coordinates are identified here for the sake of clarity in later applications.

1.2 An Illustration of the Significance of Phase Configuration

In treating the flow of a multiphase mixture, the significance of configuration of phases is seen by considering the Bernoulli equation of phases and their mixture. We note that if the Bernoulli equation for the steady flow of a phase of incompressible fluid is written as

$$\frac{\rho_k}{2} U_k^2 + P_k + \rho_k g = \text{constant}, \qquad (1.6)$$

where ρ_k is the density of phase k based on the volume of the mixture, U_k is its velocity, P_k is its pressure, and g is the appropriate component of the gravitational acceleration, the mixture will have the relation

$$\frac{\rho_m}{2} U_m^2 + \sum_k \frac{\rho_k}{2} (U_k - U_m)^2 + P_m + \rho_m g = \text{constant}, \qquad (1.7)$$

where, for the mixture,

$$\rho_m = \sum_k \rho_k, \quad \rho_m U_m = \sum_k \rho_k U_k, \qquad (1.8)$$

according to the barycentric frame of reference. Further, we have $P_m = \sum_k P_k$.

However, if one considers a highly dispersed system that can be represented by a macroscopically homogeneous mixture (Soo, 1967), we would expect that the Bernoulli equation should take the form

$$\frac{\rho_m}{2} U_m^2 + P_m + \rho_m g = \text{constant}. \qquad (1.9)$$

For the same reason as in the above, however, the Bernoulli equation of a phase k will have to take the form

$$\frac{\rho_k}{2} U_k^2 - \frac{\rho_k}{2} (U_k - U_m)^2 + P_k + \rho_k g = \text{constant}. \qquad (1.10)$$

Careful consideration of possible physical conditions will identify Eqs. 1.6 and 1.7 to be exact when phases in pure stratified flow are treated as a single system. Yet Eqs. 1.9 and 1.10 are exact for highly dispersed flow (Soo, 1981). Other flow conditions, depending on configuration of phases,

will have to be represented by Bernoulli relations in forms intermediate between the two limiting cases. The above consideration explains the inadequacy of some of the existing formulations and suggests the desirability of the rational derivation in the following chapters.

The present report focuses attention to:

1. Volume-averaged formulations including volume porosity and surface permeability for adaptation to the COMMIX code.

2. Interface transfer of mass, momentum, and energy and influence of the configuration of phases on such transfer, including the rigorous representation of the above two limiting cases - purely stratified flow and macroscopically homogeneous flow.

3. Approximations and physical basis for the present stage of COMMIX-2 code development.

2. AVERAGING RELATIONS

For the general flow system, we consider a region in Fig. 1 enclosed by a fixed control volume v and surface area A with outward normal $\underline{n}$. It includes a partially or totally (or both) immersed fixed solid phase w and a fluid mixture m with phase k and other phase or phases f (we use one phase f for representation) passing through the control volume. Phase k has a variable volume v_k with interfaces of total area A_{kf} (in v) separating phases k and f, and $\underline{n}_k$ is the outwardly directed normal from phase k on A_{kf}. Phase k also has surfaces of total area A_{wk} (in v) which is common with the immersed solids. The velocity of phase k is $\underline{U}_k$ and that of the interface $\underline{U}_{kf}$. On A_{wk} the fluid velocity and the interface velocity are zero. The surface permeabilities are associated with the entrance and exit areas A_{ek}. Overall, we have the following primary relations:

The mixture volume

$$v_m = v_k + v_f + \dots . \tag{2.1}$$

The control volume, which is also the local averaging volume

$$v = v_m + v_w. \tag{2.2}$$

The volume porosity

$$\gamma_v = v_m/v \tag{2.3}$$

is a constant for a given control volume. The volume fraction of phase k in the fluid mixture is

$$\alpha_k = v_k/v_m, \tag{2.4}$$

which is a general dependent variable.

2.1 Local Volume Average, Intrinsic Average, and Area Average

For any intensive property ψ associated with the fluid mixture m, be it a scalar, vector, or tensor, the local volume average of ψ is defined by

$$^3\langle\psi\rangle \equiv \frac{1}{v}\int_v I\,(\underline{p})\;\psi\;dv, \tag{2.5}$$

where the forward super 3 denotes volume average, $\underline{p}$ is the position vector, and the indicator function $I(\underline{p})$ is defined by

$$\left.\begin{aligned} &I(\underline{p}) = 1 \text{ for the end point of } \underline{p} \text{ in the fluid mixture} \\ &I(\underline{p}) = 0 \text{ for the end point of } \underline{p} \text{ in the solid.} \end{aligned}\right\} \tag{2.6}$$

Hence,

$$^3\langle\psi\rangle \equiv \frac{1}{v}\int_{v_m} \psi\;dv \tag{2.7}$$

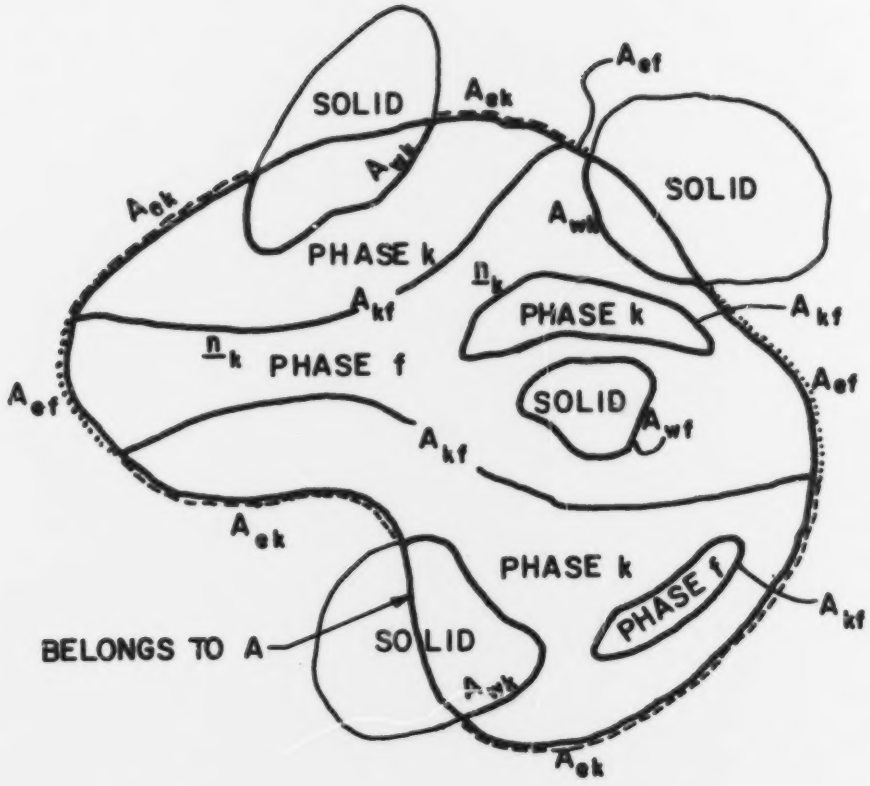

Fig. 1. MULTIPHASE SYSTEM WITH STATIONARY SOLID STRUCTURES (LOCAL AVERAGING VOLUME v WITH ENVELOPING SURFACE A)

and

$$\gamma_v = \frac{1}{v} \int_v I\ (\underline{p})\ dv = \frac{v_m}{v}, \text{ as it must be.} \tag{2.8}$$

The intrinsic local volume average of the mixture is

$$^{3i}\langle\psi\rangle = \frac{1}{v_m} \int_{v_m} \psi\ dv. \tag{2.9}$$

Further, the volume average based on the fluid mixture volume is

$$^{3m}\langle\psi\rangle = \frac{1}{v_m} \int_{v_m} \psi\ dv \tag{2.10}$$

and, we have

$$^{3}\langle\psi\rangle = \gamma_v\ ^{3i}\langle\psi\rangle \tag{2.11}$$

and

$$^{3m}\langle\psi\rangle = {}^{3i}\langle\psi\rangle. \tag{2.12}$$

For phase k, we have

$$^{3}\langle\psi_k\rangle = \frac{1}{v} \int_{v_k} \psi_k\ dv = \gamma_v \frac{1}{v_m} \int_{v_k} \psi_k\ dv, \tag{2.13}$$

$$^{3m}\langle\psi_k\rangle = \frac{1}{v_m} \int_{v_k} \psi_k\ dv = \alpha_k \frac{1}{v_k} \int_{v_k} \psi_k\ dv, \tag{2.14}$$

$$^{3i}\langle\psi_k\rangle = \frac{1}{v_k} \int_{v_k} \psi_k\ dv, \tag{2.15}$$

and, hence,

$$^{3}\langle\psi_k\rangle = \gamma_v\ ^{3m}\langle\psi_k\rangle = \gamma_v\ \alpha_k\ ^{3i}\langle\psi_k\rangle. \tag{2.16}$$

Note that while γ_v is a constant for a given v, α_k is, in general, a dependent variable.

The local area average of ψ is

$$^{2}\langle\psi\rangle \equiv \frac{1}{A}\int_{A} I\,(\underline{p})\,\psi\,dA = \frac{1}{A}\int_{A_e}\psi\,dA, \tag{2.17}$$

where the forward super 2 denotes area average, and A_e ($= A_{ek} + A_{ef}$ in Fig. 1) denotes the portion of A through which the fluid mixture flows. Setting $\psi = 1$ in Eq. 2.17, one obtains the surface permeability

$$\gamma_A = \frac{1}{A}\int_{A} I\,(\underline{p})\,dA = \frac{A_e}{A}. \tag{2.18}$$

In this report, the surface permeability always refers to entrance or exit in the enveloping surface A shown in Fig. 1, unless noted otherwise.

The intrinsic area average of the fluid mixture is

$$^{2i}\langle\psi\rangle \equiv \frac{1}{A_e}\int_{A_e}\psi\,dA \tag{2.19}$$

and

$$^{2}\langle\psi\rangle = \frac{A_e}{A}\frac{1}{A_e}\int_{A_e}\psi\,dA = \gamma_A\,{}^{2i}\langle\psi\rangle. \tag{2.20}$$

For phase k, we have

$$^{2}\langle\psi_k\rangle = \frac{1}{A}\int_{A_{ek}}\psi_k\,dA, \tag{2.21}$$

$$^{2m}\langle\psi_k\rangle = \frac{1}{A_e}\int_{A_{ek}}\psi_k\,dA, \tag{2.22}$$

$$^{2i}\langle\psi_k\rangle = \frac{1}{A_{ek}}\int_{A_{ek}}\psi_k\,dA, \tag{2.23}$$

and

$$^{2}\langle\psi_k\rangle = \gamma_A\,{}^{2m}\langle\psi_k\rangle = \gamma_A\frac{A_{ek}}{A_e}\,{}^{2i}\langle\psi_k\rangle, \tag{2.24}$$

where A_{ek}/A_e is the fraction of flow area A_e allocated to phase k.

The physical meanings of the volume averages defined in Eqs. 2.13, 2.14, and 2.15 can be seen by applying them to a fluid property, such as density of phase k. Accordingly,

$$^{3i}\langle\rho_k\rangle = v_k^{-1}\int_{v_k}\rho_k\,dv = \bar{\rho}_k, \tag{2.25}$$

where $\bar{\rho}_k$ is the mean density of the material constituting phase k,

$$^{3m}\langle\rho_k\rangle = v_m^{-1} \int_{v_k} \rho_k \, dv = \alpha_k \, ^{3i}\langle\rho_k\rangle = \alpha_k \, \bar{\rho}_k \tag{2.26}$$

and

$$^{3}\langle\rho_k\rangle = v^{-1} \int_{v_k} \rho_k \, dv = \gamma_v \, \alpha_k \, \bar{\rho}_k. \tag{2.27}$$

The meaning of A_{ek}/A_e in Eq. 2.24 can be seen by considering the area-averaged mass flux $\rho_k \underline{U}_k$ associated with the local averaging volume $\Delta x \Delta y \Delta z$. Consider, for example, the mass flow rate of a mixture of two phases k and f through area $\Delta A_{e,x}$ which may be a portion of $\Delta A_x (= \Delta y \Delta z)$:

$$(\rho_k U_{kx} + \rho_f U_{fx}) \Delta A_{e,x} = \bar{\rho}_k U_{kx} \Delta A_{ek,x} + \bar{\rho}_f U_{fx} \Delta A_{ef,x}. \tag{2.28}$$

But $\rho_k = \alpha_k \bar{\rho}_k$, $\rho_f = \alpha_f \bar{\rho}_f$; hence,

$$\frac{\Delta A_{ek,x}}{\Delta A_{e,x}} = \alpha_k, \quad \text{and} \quad \frac{\Delta A_{ef,x}}{\Delta A_{e,x}} = \alpha_f = 1 - \alpha_k. \tag{2.29}$$

Using the same reasoning, one may show that

$$\frac{\Delta A_{ek,y}}{\Delta A_{e,y}} = \frac{\Delta A_{ek,z}}{\Delta A_{e,z}} = \alpha_k \tag{2.30a}$$

and

$$\frac{\Delta A_{ef,y}}{\Delta A_{e,y}} = \frac{\Delta A_{ef,z}}{\Delta A_{e,z}} = \alpha_f. \tag{2.30b}$$

Note that α_k in the above relations is consistent with the specification of the length-scale relationship given later by Eq. 2.46. In a strict sense, the above relation is valid for approximating a homogeneous nonstructured medium (Whitaker, 1969).

We shall have occasion to relate a volume-averaged flow or flux related quantity $\underline{\psi}_k$ to its area-averaged counterpart. In a Cartesian system of coordinates, the relationship may be developed as follows.

$$^{3}\langle\underline{\psi}_k\rangle = \frac{1}{v} \int_{v_k} \underline{\psi}_k \, dv = \frac{1}{\Delta x} \int_{x' = -\frac{\Delta x}{2}}^{x' = +\frac{\Delta x}{2}} \left(\frac{1}{\Delta y \Delta z} \int_{A_k} \underline{\psi}_k \, dA \right) dx'. \tag{2.31}$$

But

$$\frac{1}{\Delta y \Delta z} \int_{A_k} \underline{\psi}_k \, dA = \frac{\Delta A'_{e,x}}{\Delta A_x} \frac{\Delta A'_{ek,x}}{\Delta A'_{e,x}} \, {}^{2i(x)}\langle \underline{\psi}_k \rangle', \tag{2.32}$$

where all primed quantities refer to local values inside v. The forward superscript 2i(x) denotes intrinsic area average, the normal to the area being in the x direction. Now,

$$\frac{\Delta A'_{e,x}}{\Delta A_x} = \gamma'_{Ax} \tag{2.33}$$

is the local surface permeability (interior of v) associated with fluid flow as a mixture. From Eqs. 2.29 and 2.30, we have

$$\frac{\Delta A'_{ek,x}}{\Delta A'_{e,x}} = \frac{\Delta A'_{ek,y}}{\Delta A'_{e,y}} = \frac{\Delta A'_{ek,z}}{\Delta A'_{e,z}} = \alpha_k. \tag{2.34}$$

Consider $\underline{\psi}_k = \underline{i}\,\psi_{kx} + \underline{j}\,\psi_{ky} + \underline{k}\,\psi_{kz}$; general averaging gives (Sha and Chao, 1981)

$${}^{3}\langle \psi_{kx} \rangle = {}^{1(x)}\left\langle \gamma'_{Ax}\, \alpha_k\, {}^{2i(x)}\langle \psi_{kx} \rangle' \right\rangle, \tag{2.35}$$

$${}^{3}\langle \psi_{ky} \rangle = {}^{1(y)}\left\langle \gamma'_{Ay}\, \alpha_k\, {}^{2i(y)}\langle \psi_{ky} \rangle' \right\rangle, \tag{2.36}$$

and

$${}^{3}\langle \psi_{kz} \rangle = {}^{1(z)}\left\langle \gamma'_{Az}\, \alpha_k\, {}^{2i(z)}\langle \psi_{kz} \rangle' \right\rangle, \tag{2.37}$$

where ${}^{1(\ )}\langle\ \rangle$ denotes segmental average.

Consequently,

$$\begin{aligned} {}^{3}\langle \underline{\psi}_k \rangle = {} & \underline{i}\, {}^{1(x)}\left\langle \gamma'_{Ax}\, \alpha_k\, {}^{2i(x)}\langle \psi_{kx} \rangle' \right\rangle \\ & + \underline{j}\, {}^{1(y)}\left\langle \gamma'_{Ay}\, \alpha_k\, {}^{2i(y)}\langle \psi_{ky} \rangle' \right\rangle \\ & + \underline{k}\, {}^{1(z)}\left\langle \gamma'_{Az}\, \alpha_k\, {}^{2i(z)}\langle \psi_{kz} \rangle' \right\rangle. \end{aligned} \tag{2.38}$$

While the segmental averages given in Eqs 2.35-2.38 are mathematically correct, they are not amenable to computation. Furthermore, they are incompatible with the meaning of $\underline{U}_k$ (when $\underline{\psi}_k = \underline{U}_k$) already explained. Should Eq. 2.38 be used for ${}^{3}\langle \underline{U}_k \rangle$, the latter represents the volume average of the interstitial velocites of phase k. Accordingly, the intrinsic area average of

ψ_{kx} associated with entrance or exit area $A_{ek,x}$ is related to the segmental average as

$$\gamma_{Ax}\, \alpha_k\, {}^{2i(x)}\langle\psi_{kx}\rangle \equiv {}^{1(x)}\left\langle \gamma'_{Ax}\, \alpha_k\, {}^{2i(x)}\langle\psi_{kx}\rangle' \right\rangle \tag{2.39}$$

and similar relationships for the y and z directions. Thus, we have the following relation:

$$\begin{aligned} {}^{3}\langle\underline{\psi}_k\rangle = {} & \underline{i}\, \gamma_{Ax}\, \alpha_k\, {}^{2i(x)}\langle\psi_{kx}\rangle \\ & + \underline{j}\, \gamma_{Ay}\, \alpha_k\, {}^{2i(y)}\langle\psi_{ky}\rangle \\ & + \underline{k}\, \gamma_{Az}\, \alpha_k\, {}^{2i(z)}\langle\psi_{kz}\rangle, \end{aligned} \tag{2.40}$$

for which we write, for compactness,

$$\nabla \cdot {}^{3}\langle\underline{\psi}_k\rangle = \nabla \cdot \gamma_A\, \alpha_k\, {}^{2i}\langle\underline{\psi}_k\rangle, \tag{2.41}$$

where $\underline{\psi}_k$ is any flow- or flux-related quantity. This result is also the direct consequence of Eq. 2.45 in the following section.

2.2 Local Averaging Theorems

The volume averages of derivatives of ψ_k are given by the general transport theorem as (Slattery, 1967; Whitaker, 1967, 1969)

$${}^{3}\langle\nabla\psi_k\rangle = \nabla\, {}^{3}\langle\psi_k\rangle + v^{-1} \int_{A_{kf}} \psi_k\, \underline{n}_k\, dA + v^{-1} \int_{A_{wk}} \psi_k\, \underline{n}_k\, dA, \tag{2.42}$$

$${}^{3}\langle\nabla \cdot \underline{\psi}_k\rangle = \nabla \cdot {}^{3}\langle\underline{\psi}_k\rangle + v^{-1} \int_{A_{kf}} \underline{\psi}_k \cdot \underline{n}_k\, dA + v^{-1} \int_{A_{wk}} \underline{\psi}_k \cdot \underline{n}_k\, dA, \tag{2.43}$$

and

$${}^{3}\langle\partial\psi_k/\partial t\rangle = \partial\, {}^{3}\langle\psi_k\rangle/\partial t - v^{-1} \int_{A_{kf}} \psi_k \underline{U}_{kf} \cdot \underline{n}_k\, dA \tag{2.44}$$

since $\underline{U}_{kf}$ on A_{wk} is zero. In Eq. 2.44 $\underline{U}_{kf} \cdot \underline{n}_k$ is the speed of displacement of the interface. In addition,

$$\nabla \cdot {}^{3}\langle\underline{\psi}_k\rangle = v^{-1} \int_{A_e} \underline{\psi}_k \cdot \underline{n}_k\, dA. \tag{2.45}$$

Note that these averaging relations are subject to the restriction that (Whitaker, 1969)

(Characteristic length of pores and phases)

<< (characteristic length of averaging volume)

<< (characteristic length of the physical system). (2.46)

The averaging volume under consideration, therefore, cannot be arbitrarily small or become infinitesimal.

For $\psi_k = 1$, Eq. 2.44 together with Eq. 2.13 gives

$$\gamma_v \left(\partial \alpha_k / \partial t\right) = v^{-1} \int_{A_{kf}} \underline{U}_{kf} \cdot \underline{n}_k \, dA \tag{2.47}$$

and Eq. 2.42 gives

$$\nabla \gamma_v \alpha_k = - v^{-1} \int_{A_{kf}} \underline{n}_k \, dA - v^{-1} \int_{A_{wk}} \underline{n}_k \, dA. \tag{2.48}$$

Summing up for all fluid phases, we have from Eq. 2.48

$$\nabla \gamma_v = - v^{-1} \left(\sum_k \int_{A_{kf}} \underline{n}_k \, dA + \sum_k \int_{A_{wk}} \underline{n}_k \, dA \right) \tag{2.49}$$

since $\sum_k \alpha_k = 1$. For a system consisting of a dispersed phase and a continuous phase, A_{kf} is an aggregate of closed surfaces and hence $\sum_k \int_{A_k} \underline{n}_k \, dA = 0$. In fact, it must necessarily vanish since γ_v is not physically related to A_{kf}. Furthermore, for all <u>interior</u> solids, $\sum_k \int_{A_{wk}} \underline{n}_k \, dA = 0$. Consequently,

$$\nabla \gamma_v = 0 \tag{2.50}$$

if the bounding surface A of the averaging volume v contains no solids. It is noteworthy that Eq. 2.50 remains valid if A cuts across the solids but the solids are uniformly distributed over A. While Eq. 2.50 is not generally true, $\nabla \gamma_v$ can be made to vanish or be small by a judicious selection of the computational cells in numerical analysis.

For any flow- or flux-related quantity, $\underline{\psi}_k$, we have on the one hand,

$$\nabla \cdot {}^{3}\langle \underline{\psi}_k \rangle = \nabla \cdot v^{-1} \int_{v_k} \underline{\psi}_k \, dv = \nabla \cdot \gamma_A \alpha_k {}^{2i}\langle \underline{\psi}_k \rangle$$

$$= \nabla \cdot \gamma_A {}^{2m}\langle \underline{\psi}_k \rangle \tag{2.51}$$

as has been previously shown in Eq. 2.41. On the other hand, for any scalar intensive property

$$\nabla\, {}^{3}\langle\psi_k\rangle = \nabla\, \gamma_v\, \alpha_k\, {}^{3i}\langle\psi_k\rangle = \nabla\, \gamma_v\, {}^{3m}\langle\psi_k\rangle. \tag{2.52}$$

3. DYNAMIC EQUATIONS OF A PHASE, INTERFACE AND AVERAGES

This section presents the phasic equations of continuum mechanics as applied to a pure phase, the dynamic equations of the interface, and their local volume averages. The notations are chosen to minimize the complexity in subsequent derivations and to provide the shortest linkage between terms in the basic equations and experimental facts. Even though the expressions indicate volume averaging, time averaging after volume averaging can be made subsequently to reduce high-frequency fluctuations in transport parameter and additional correlations resulting from turbulence, diffusion, etc.

3.1 Dynamics of a Phase and an Interface

The phasic equations of continuum mechanics as applied to a pure phase are well known. In a pure phase k, the equations of continuity, momentum, and energy are

$$(\partial \rho_k / \partial t) + \nabla \cdot (\rho_k \underline{U}_k) = 0, \tag{3.1}$$

$$(\partial \rho_k \underline{U}_k / \partial t) + \nabla \cdot (\rho_k \underline{U}_k \underline{U}_k) = - \nabla P_k + \nabla \cdot \underline{\tau}_k + \rho_k \underline{f}, \tag{3.2}$$

and

$$(\partial \rho_k E_k / \partial t) + \nabla \cdot (\rho_k \underline{U}_k E_k) = - \nabla \cdot \underline{U}_k P_k + \nabla \cdot (\underline{U}_k \cdot \underline{\tau}_k) + \rho_k \underline{U}_k \cdot \underline{f} - \nabla \cdot \underline{J}_{qk} + J_{Ek}, \tag{3.3}$$

where ρ_k is the density of fluid in pure phase k, $\underline{U}_k$ is its velocity, P_k is the static pressure inside phase k, $\underline{f}$ is the field force per unit mass, which is taken to be a constant in the present study, $\underline{\tau}_k$ is the shear stress tensor, the total energy per unit mass is $E_k = u_k + U_k^2/2$, u_k is the internal energy, $\underline{J}_{qk}$ is the heat-flux vector, and J_{Ek} is the heat source inside phase k. When applied to a multiphase system (Fig. 1), we have, in addition to Eqs. 3.1-3.3, the interface boundary conditions, identifying the velocity of the interface $\underline{U}_{kf}$, pressure P_{kf}, and temperature T_{kf}. Proper averaging is, therefore, needed to avoid solving the multiphase flow problem as a deterministic multi-boundary problem with the interface relations of mass, momentum, and energy as boundary conditions.

The simplest case of the interface is one of zero thickness. The mass, momentum, and energy balances at the interface A_{kf} (between phases k and f, Fig. 1) are represented by

$$\rho_k (\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k + \rho_f (\underline{U}_f - \underline{U}_{fk}) \cdot \underline{n}_f = 0, \tag{3.4}$$

$$- \nabla_{kf} \sigma_{kf} + 2\sigma_{kf} H_{kf} \underline{n}_k - \rho_k \underline{U}_k (\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k - \rho_f \underline{U}_f (\underline{U}_f - \underline{U}_{fk}) \cdot \underline{n}_f$$

$$+ \left(- \underline{\underline{I}}\, P_k + \underline{\underline{\tau}}_k\right) \cdot \underline{n}_k + \left(- \underline{\underline{I}} P_f + \underline{\underline{\tau}}_f\right) \cdot \underline{n}_f = 0, \tag{3.5}$$

and

$$\rho_k E_k \left(\underline{U}_k - \underline{U}_{kf}\right) \cdot \underline{n}_k + \underline{J}_{qk} \cdot \underline{n}_k$$

$$+ \rho_f E_f \left(\underline{U}_f - \underline{U}_{fk}\right) \cdot \underline{n}_f + \underline{J}_{qf} \cdot \underline{n}_f$$

$$+ \underline{U}_k \cdot \left(- \underline{\underline{I}} P_k + \underline{\underline{\tau}}_k\right) \cdot \underline{n}_k + U_f \cdot \left(- \underline{\underline{I}} P_f + \underline{\underline{\tau}}_f\right) \cdot \underline{n}_f = 0, \tag{3.6}$$

where $\underline{n}_k$ is the directional normal outward from phase k and also directed along the mean radius of curvature H_{kf}^{-1}, σ_{kf} is the interfacial tension, ∇_{kf} is the surface gradient operator, and $\underline{\underline{I}}$ is the unitary tensor. In Eq. 3.6, the energy due to surface tension and the corresponding dissipation are neglected (Sha and Slattery, 1980). Note that $\underline{U}_{kf} = \underline{U}_{fk}$; H_{kf}^{-1} is positive when pointing outward.

3.2 Volume-Averaged Dynamic Equations of a Phase and its Interfaces

When the local volume-averaged relations are applied to the terms in the continuity equation (Eq. 3.1), we get from Eqs. 2.43, 2.44, and 2.16

$$^3\langle \partial \rho_k / \partial t \rangle = \gamma_v \left(\partial\, ^{3m}\langle \rho_k \rangle / \partial t\right) - v^{-1} \int_{A_{kf}} \rho_k \underline{U}_{kf} \cdot \underline{n}_k \, dA \tag{3.7}$$

and

$$^3\langle \nabla \cdot (\rho_k \underline{U}_k) \rangle = \nabla \cdot {}^3\langle \rho_k \underline{U}_k \rangle + \bar{v}^{-1} \int_{A_{kf}} \rho_k \underline{U}_k \cdot \underline{n}_k \, dA + v^{-1} \int_{A_{wk}} \rho_k \underline{U}_k \cdot \underline{n}_k \, dA$$

$$= \nabla \cdot \gamma_A \, {}^{2m}\langle \rho_k \underline{U}_k \rangle + v^{-1} \int_{A_{kf}} \rho_k \underline{U}_k \cdot \underline{n}_k \, dA, \tag{3.8}$$

since $\underline{U}_k = 0$ on A_{wk}. In writing Eq. 3.8, we have used the relation given by Eq. 2.41. Further,

$$\nabla \cdot \gamma_A \, {}^{2m}\langle \rho_k \underline{U}_k \rangle = \nabla \cdot \gamma_A \, \alpha_k \, {}^{2i}\langle \rho_k \underline{U}_k \rangle. \tag{3.9}$$

Hence, the continuity equation takes the form

$$\gamma_v \left(\partial\, ^{3m}\langle \rho_k \rangle / \partial t\right) + \nabla \cdot \gamma_A \, {}^{2m}\langle \rho_k \underline{U}_k \rangle$$

$$= - v^{-1} \int_{A_{kf}} \rho_k (\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k \, dA = \gamma_v \Gamma_k, \tag{3.10}$$

or, in terms of intrinsic averages,

$$\gamma_v \left(\partial \, \alpha_k \, {}^{3i}\langle \rho_k \rangle / \partial t \right) + \nabla \cdot \gamma_A \, \alpha_k \, {}^{2i}\langle \rho_k \underline{U}_k \rangle = \gamma_v \Gamma_k. \tag{3.11}$$

In Eqs. 3.10 and 3.11, Γ_k is the net rate of generation of phase k over A_{kf} per unit volume of the fluid mixture, and γ_A is the surface permeability in the form of vector coefficients in Eq. 1.4.

The interface balance relation (Eq. 3.4) now gives

$$v^{-1} \int_{A_{kf}} \rho_k (\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k \, dA = - v^{-1} \int_{A_{kf}} \rho_f (\underline{U}_f - \underline{U}_{kf}) \cdot \underline{n}_f \, dA$$

$$= \gamma_v \Gamma_k = - \gamma_v \Gamma_f. \tag{3.12}$$

When the local volume averaging is applied to the terms of the momentum equation (Eq. 3.2), Eqs. 2.42, 2.43, 2.44, 2.51, and 2.52 give

$${}^{3}\langle \partial \rho_k \underline{U}_k / \partial t \rangle = \gamma_v \left(\partial \, {}^{3m}\langle \rho_k \underline{U}_k \rangle / \partial t \right) - v^{-1} \int_{A_{kf}} \rho_k \, \underline{U}_k \underline{U}_{kf} \cdot \underline{n}_k \, dA, \tag{3.13}$$

$${}^{3}\langle \nabla \cdot \rho_k \underline{U}_k \underline{U}_k \rangle = \nabla \cdot \gamma_A \, {}^{2m}\langle \rho_k \underline{U}_k \underline{U}_k \rangle + v^{-1} \int_{A_{kf}} \rho_k \, \underline{U}_k \underline{U}_k \cdot \underline{n}_k \, dA, \tag{3.14}$$

$${}^{3}\langle \nabla P_k \rangle = \nabla \, {}^{3}\langle P_k \rangle + v^{-1} \int_{A_{kf}} P_k \, \underline{n}_k \, dA + v^{-1} \int_{A_{wk}} P_k \, \underline{n}_k \, dA \tag{3.15}$$

$$= \nabla \, \gamma_v \, {}^{3m}\langle P_k \rangle + v^{-1} \int_{A_{kf}} P_k \underline{n}_k \, dA + v^{-1} \int_{A_{wk}} P_k \underline{n}_k \, dA, \tag{3.16}$$

and

$${}^{3}\langle \nabla \cdot \underline{\tau}_k \rangle = \nabla \cdot \gamma_A \, {}^{2m}\langle \underline{\tau}_k \rangle + v^{-1} \int_{A_{kf}} \underline{\tau}_k \cdot \underline{n}_k \, dA + v^{-1} \int_{A_{wk}} \underline{\tau}_k \cdot \underline{n}_k \, dA. \tag{3.17}$$

The choice between ${}^{2m}\langle \underline{\tau}_k \rangle$ and ${}^{2i}\langle \underline{\tau}_k \rangle$ would depend on the configuration of phases; the former is appropriate for dispersed flows and refers to shear stresses of phase k in the mixture (Soo, 1968), while the latter is appropriate for stratified flows.

When Eqs. 3.13-3.17 are combined separately, we note that the following terms can be grouped as

(1)

$$v^{-1} \int_{A_{wk}} (-P_k \underline{n}_k + \underline{\tau}_k \cdot \underline{n}_k) dA = - v^{-1} \int_{v_k} \underline{R}_k \, dv = - \gamma_v^{3m} \langle \underline{R}_k \rangle, \quad (3.18)$$

where $\underline{R}_k$ is the resistance force per unit volume of the fluid mixture exerted by the fixed dispersed solid on phase k;

(2)

$$v^{-1} \int_{A_{kf}} (-P_k \underline{n}_k + \underline{\tau}_k \cdot \underline{n}_k) dA \quad (3.19)$$

gives the interfacial drag and the diffusive force (Sha and Soo, 1979; Soo, 1978, 1981) on phase k by phase f in the mixture;

(3)

$$v^{-1} \int_{A_{kf}} \rho_k \underline{U}_k (\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k \, dA$$

$$= v^{-1} \int_{A_{kf}} \rho_k (\underline{U}_k - \underline{U}_{kf})(\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k \, dA + v^{-1} \int_{A_{kf}} \rho_k \underline{U}_{kf} (\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k \, dA$$

$$= v^{-1} \int_{A_{kf}} \rho_k (\underline{U}_k - \underline{U}_{kf})(\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k \, dA - \gamma_v \Gamma_k \underline{U}'_s, \quad (3.20)$$

where $\underline{U}'_s$ is defined by

$$\underline{U}'_s = \int_{A_{kf}} \rho_k \underline{U}_{kf} (\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k \, dA \Big/ \int_{A_{kf}} \rho_k (\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k \, dA, \quad (3.21)$$

which is a mean interface velocity at which phase k is generated or removed over A_{kf}.

Note that the transfer integral in Eq. 3.20 does not necessarily go to zero when $\Gamma_k = 0$ (Eq. 3.10) because of the condition given in Eq. 2.46 and the local averaging volume v should not be made arbitrarily small. It is conceivable that condensation may occur over a portion of A_{kf} and evaporation over the remaining portion. Thus, the integral may vanish, i.e., $\Gamma_k = 0$. It does not necessarily follow that, locally, $(\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k = 0$.

The resultant momentum equation takes the form

$$\gamma_v \frac{\partial}{\partial t}{}^{3m} \langle \rho_k \underline{U}_k \rangle + \nabla \cdot \gamma_A{}^{2m} \langle \rho_k \underline{U}_k \underline{U}_k \rangle$$

$$= - \nabla \gamma_v \, {}^{3m}\langle P_k \rangle + \nabla \cdot \gamma_A \, {}^{2m}\langle \underline{\underline{\tau}}_k \rangle + \gamma_v \left({}^{3m}\langle \rho_k \rangle \, \underline{f} - {}^{3m}\langle \underline{R}_k \rangle \right)$$

$$+ v^{-1} \int_{A_{kf}} \left(-P_k \underline{n}_k + \underline{\underline{\tau}}_k \cdot \underline{n}_k \right) dA - v^{-1} \int_{A_{kf}} \rho_k \underline{U}_k \left(\underline{U}_k - \underline{U}_{kf} \right) \cdot \underline{n}_k \, dA, \quad (3.22)$$

or, in terms of intrinsic averages,

$$\gamma_v \frac{\partial}{\partial t} \left(\alpha_k \, {}^{3i}\langle \rho_k \underline{U}_k \rangle \right) + \nabla \cdot \gamma_A \alpha_k \, {}^{2i}\langle \rho_k \underline{U}_k \underline{U}_k \rangle$$

$$= - \nabla \gamma_v \alpha_k \, {}^{3i}\langle P_k \rangle + \nabla \cdot \gamma_A \, \alpha_k \, {}^{2i}\langle \underline{\underline{\tau}}_k \rangle + \gamma_v \alpha_k \left({}^{3i}\langle \rho_k \rangle \, \underline{f} - {}^{3i}\langle \underline{R}_k \rangle \right)$$

$$+ v^{-1} \int_{A_{kf}} \left(-P_k \underline{n}_k + \underline{\underline{\tau}}_k \cdot \underline{n}_k \right) dA - v^{-1} \int_{A_{kf}} \rho_k \underline{U}_k \left(\underline{U}_k - \underline{U}_{kf} \right) \cdot \underline{n}_k \, dA. \quad (3.23)$$

The last term in Eq. 3.22 or 3.23 may be replaced by its equivalent, Eq. 3.20.

The first term on the right-hand side of Eq. 3.23 may be written as, with Eq. 2.50,

$$- \nabla \gamma_v \alpha_k \, {}^{3i}\langle P_k \rangle = - \gamma_v \nabla \alpha_k \, {}^{3i}\langle P_k \rangle. \quad (3.24)$$

Then, Eq. 3.23 may be rewritten as

$$\gamma_v \frac{\partial}{\partial t} \left(\alpha_k \, {}^{3i}\langle \rho_k \underline{U}_k \rangle + \nabla \cdot \gamma_A \alpha_k \, {}^{2i}\langle \rho_k \underline{U}_k \underline{U}_k \rangle \right.$$

$$= - \gamma_v \nabla \alpha_k \, {}^{3i}\langle P_k \rangle + \nabla \cdot \gamma_A \, \alpha_k \, {}^{2i}\langle \underline{\underline{\tau}}_k \rangle + \gamma_v \alpha_k \left({}^{3i}\langle \rho_k \rangle \, \underline{f} - {}^{3i}\langle \underline{R}_k \rangle \right)$$

$$+ v^{-1} \int_{A_{kf}} \left(-P_k \underline{\underline{I}} + \underline{\underline{\tau}}_k \right) \cdot \underline{n}_k \, dA - v^{-1} \int_{A_{kf}} \rho_k \underline{U}_k \left(\underline{U}_k - \underline{U}_{kf} \right) \cdot \underline{n}_k \, dA. \quad (3.25)$$

For a pure phase k, $\alpha_k = 1$, $A_{kf} = 0$, and in the absence of any motion, Eq. 3.25 gives

$$\nabla \, {}^{3i}\langle P_k \rangle - {}^{3i}\langle \rho_k \rangle \, \underline{f} = 0 \quad (3.26)$$

or

$$\nabla P = \overline{\rho}_k \underline{g} \quad \text{with } \underline{f} = \underline{g}, \quad (3.27)$$

thus satisfying the basic relation of hydrostatics.

The first two terms in Eq. 3.5 give rise to capillary pressure difference across the interface. Consequently, we define an average capillary pressure difference as

$$\int_{A_{kf}} \left({}^{3i}\langle P_{ck} \rangle - {}^{3i}\langle P_{cf} \rangle \right) \underline{n}_k \, dA = \int_{A_{kf}} \left(-\nabla_{kf} \sigma_{kf} + 2\sigma_{kf} H_{kf} \underline{n}_k \right) dA. \tag{3.28}$$

The local volume-averaged momentum balance for the interface A_{kf} is, from Eqs. 3.5 and 3.28

$$v^{-1} \int_{A_{kf}} \rho_k \underline{U}_k (\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k \, dA + v^{-1} \int_{A_{kf}} \left[(P_k - {}^{3i}\langle P_{ck} \rangle) \underline{\underline{I}} - \underline{\tau}_k \right] \cdot \underline{n}_k \, dA$$

$$= - v^{-1} \int_{A_{kf}} \rho_f \underline{U}_k (\underline{U}_f - \underline{U}_{kf}) \cdot \underline{n}_f \, dA - v^{-1} \int_{A_{kf}} \left[(P_f - {}^{3i}\langle P_{cf} \rangle) \underline{\underline{I}} - \underline{\tau}_f \right] \cdot \underline{n}_f \, dA. \tag{3.29}$$

Generally speaking, the capillary pressure difference can be neglected for bubbles or droplets of diameters greater than 0.1 mm (Campbell and Pitcher, 1958).

When the energy equation (Eq. 3.6) is averaged, the terms are

$${}^{3}\langle \partial \rho_k E_k / \partial t \rangle = \gamma_v \, \partial \, {}^{3m}\langle \rho_k E_k \rangle / \partial t - v^{-1} \int_{A_{kf}} \rho_k E_k \underline{U}_{kf} \cdot \underline{n}_k \, dA, \tag{3.30}$$

$${}^{3}\langle \nabla \cdot (\rho_k \underline{U}_k E_k) \rangle = \nabla \cdot \gamma_A \, {}^{2m}\langle \rho_k \underline{U}_k E_k \rangle + v^{-1} \int_{A_{kf}} \rho_k \underline{U}_k E_k \cdot \underline{n}_k \, dA, \tag{3.31}$$

$${}^{3}\langle \nabla \cdot (\underline{U}_k P_k) \rangle = \nabla \cdot \gamma_A \, {}^{2m}\langle \underline{U}_k P_k \rangle + v^{-1} \int_{A_{kf}} \underline{U}_k P_k \cdot \underline{n}_k \, dA, \tag{3.32}$$

$${}^{3}\langle \nabla \cdot \underline{U}_k \cdot \underline{\tau}_k \rangle = \nabla \cdot \gamma_A \, {}^{2m}\langle \underline{U}_k \cdot \underline{\tau}_k \rangle + v^{-1} \int_{A_{kf}} \underline{U}_k \cdot \underline{\tau}_k \cdot \underline{n}_k \, dA, \tag{3.33}$$

$${}^{3}\langle \rho_k \underline{U}_k \rangle \cdot \underline{f} = \gamma_v \, {}^{3m}\langle \rho_k \underline{U}_k \rangle \cdot \underline{f}, \tag{3.34}$$

and

$${}^{3}\langle \nabla \cdot \underline{J}_{qk} \rangle = \nabla \cdot \gamma_A \, {}^{2m}\langle \underline{J}_{qk} \rangle + v^{-1} \int_{A_{kf}} \underline{J}_{qk} \cdot \underline{n}_k \, dA + v^{-1} \int_{A_{wk}} \underline{J}_{qk} \cdot \underline{n}_k \, dA$$

$$= \nabla \cdot \gamma_A \, {}^{2m}\langle \underline{J}_{qk} \rangle - \gamma_v \, {}^{3m}\langle \dot{Q}_{kf} \rangle - \gamma_v \, {}^{3m}\langle \dot{Q}_{wk} \rangle, \tag{3.35}$$

with

$$\gamma_v \, {}^{3m}\langle \dot{Q}_{kf} \rangle = - v^{-1} \int_{A_{kf}} \underline{J}_{qk} \cdot \underline{n}_k \, dA \tag{3.36}$$

and

$$\gamma_v \, {}^{3m}\langle \dot{Q}_{wk} \rangle = \gamma_v \alpha_k \, {}^{3i}\langle \dot{Q}_{wk} \rangle = - v^{-1} \int_{A_{wk}} \underline{J}_{qk} \cdot \underline{n}_k \, dA, \tag{3.37}$$

where ${}^{3m}\langle \dot{Q}_{kf} \rangle$ is the rate of heat transport to phase k across A_{kf} per unit volume of the fluid mixture, and ${}^{3i}\langle \dot{Q}_{wk} \rangle$ is the rate of heat transfer from solid wall A_{wk} to phase k per unit volume of phase k. ${}^{3m}\langle \dot{Q}_{kf} \rangle$ would depend on the physical and chemical processes occurring in A_{kf} as well as its configuration.

And finally,

$$v^{-1} \int_{A_{kf}} \rho_k E_k (\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k \, dA$$

$$= v^{-1} \int_{A_{kf}} \rho_k (E_k - E_{kf})(\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k \, dA - \gamma_v \Gamma_k E'_s, \tag{3.38}$$

where E'_s is the mean interface total energy given by

$$E'_s = \frac{\int_{A_{kf}} \rho_k E_{kf} (\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k \, dA}{\int_{A_{kf}} \rho_k (\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k \, dA} . \tag{3.39}$$

The averaged total energy equation takes the form

$$\begin{aligned}
&\gamma_v \frac{\partial}{\partial t} \, {}^{3m}\langle \rho_k E_k \rangle + \nabla \cdot \gamma_A \, {}^{2m}\langle \rho_k \underline{U}_k E_k \rangle \\
&= - \nabla \cdot \gamma_A \, {}^{2m}\langle \underline{U}_k P_k \rangle + \nabla \cdot \gamma_A \, {}^{2m}\langle \underline{U}_k \cdot \underline{\tau}_k \rangle - \nabla \cdot \gamma_A \, {}^{2m}\langle \underline{J}_{qk} \rangle \\
&+ \gamma_v \left({}^{3m}\langle \rho_k \underline{U}_k \rangle \cdot \underline{f} + {}^{3m}\langle J_{Ek} \rangle + {}^{3m}\langle \dot{Q}_{kf} \rangle + {}^{3m}\langle \dot{Q}_{wk} \rangle \right) \\
&+ v^{-1} \int_{A_{kf}} (-P_k \underline{U}_k + \underline{\tau}_k \cdot \underline{U}_k) \cdot \underline{n}_k \, dA - v^{-1} \int_{A_{kf}} \rho_k E_k (\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k \, dA.
\end{aligned} \tag{3.40}$$

Or, in terms of intrinsic averages,

$$\gamma_v \frac{\partial}{\partial t}\left(\alpha_k \, {}^{3i}\langle \rho_k E_k \rangle\right) + \nabla \cdot \gamma_A \alpha_k \, {}^{2i}\langle \rho_k \underline{U}_k E_k \rangle$$

$$= -\nabla \cdot \gamma_A \alpha_k \, {}^{2i}\langle \underline{U}_k P_k \rangle + \nabla \cdot \gamma_A \alpha_k \, {}^{2i}\langle \underline{U}_k \cdot \underline{\tau}_k \rangle - \nabla \cdot \gamma_A \alpha_k \, {}^{2i}\langle \underline{J}_{qk} \rangle$$

$$+ \gamma_v \alpha_k \left({}^{3i}\langle \rho_k \underline{U}_k \rangle \cdot \underline{f} + {}^{3i}\langle J_{Ek} \rangle + {}^{3i}\langle \dot{Q}_{wk} \rangle\right) + \gamma_v \, {}^{3m}\langle \dot{Q}_{kf} \rangle$$

$$+ v^{-1} \int_{A_{kf}} \left(-P_k \underline{U}_k + \underline{\tau}_k \cdot \underline{U}_k\right) \cdot \underline{n}_k \, dA - v^{-1} \int_{A_{kf}} \rho_k E_k \left(\underline{U}_k - \underline{U}_{kf}\right) \cdot \underline{n}_k \, dA. \tag{3.41}$$

The last term in Eq. 3.40 or 3.41 may be replaced by its equivalent, Eq. 3.39. Again, we note that Eq. 3.40 is appropriate for a dispersed system (Soo, 1968), while Eq. 3.41 is suitable for stratified flows.

Ignoring capillary energy, the local volume-averaged total energy balance for the interfaces A_{kf} is, from Eq. 3.6

$$v^{-1} \int_{A_{kf}} \rho_k E_k \left(\underline{U}_k - \underline{U}_{kf}\right) \cdot \underline{n}_k \, dA + v^{-1} \int_{A_{kf}} \underline{J}_{qk} \cdot \underline{n}_k \, dA$$

$$+ v^{-1} \int_{A_{kf}} \underline{U}_k \cdot \left(-P_k \underline{I} + \underline{\tau}_k\right) \cdot \underline{n}_k \, dA + v^{-1} \int_{A_{kf}} \rho_f E_f \left(\underline{U}_f - \underline{U}_{kf}\right) \cdot \underline{n}_k \, dA$$

$$+ v^{-1} \int_{A_{kf}} \underline{J}_{qf} \cdot \underline{n}_f \, dA + v^{-1} \int_{A_{kf}} \underline{U}_f \cdot \left(-P_f \underline{I} + \underline{\tau}_f\right) \cdot \underline{n}_f \, dA = 0. \tag{3.42}$$

3.3 Volume-Averaged Energy Equation in Terms of Internal Energy

Alternately, the energy equation can be expressed in terms of internal energy instead of total energy. For phase k, we have

$$\frac{\partial \rho_k u_k}{\partial t} + \nabla \cdot \left(\rho_k \underline{U}_k u_k\right) = -P_k \nabla \cdot \underline{U}_k - \nabla \cdot \underline{J}_{qk} + J_{Ek} - \underline{\tau}_k : \nabla \underline{U}_k. \tag{3.43}$$

The double dot in the last term denotes the scalar product of two second-order tensors and is usually represented as Φ_k, the dissipation rate per unit volume of phase k.

The local volume average of Eq. 3.43 is

$$\gamma_v \frac{\partial}{\partial t} \, {}^{3m}\langle \rho_k u_k \rangle + \nabla \cdot \gamma_A \, {}^{2m}\langle \rho_k \underline{U}_k u_k \rangle$$

$$= -\gamma_v \, {}^{3m}\langle P_k \nabla \cdot \underline{U}_k \rangle - \nabla \cdot \gamma_A \, {}^{2m}\langle \underline{J}_{qk} \rangle$$

$$+ \gamma_v \left({}^{3m}\langle J_{Ek} \rangle + {}^{3m}\langle \phi_k \rangle + {}^{3m}\langle \dot{Q}_{wk} \rangle \right)$$

$$- v^{-1} \int_{A_{kf}} \underline{J}_{qk} \cdot \underline{n}_k \, dA - v^{-1} \int_{A_{kf}} \rho_k u_k (\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k \, dA. \qquad (3.44)$$

Alternatively, in terms of intrinsic averages, it is

$$\gamma_v \frac{\partial}{\partial t} \left(\alpha_k \, {}^{3i}\langle \rho_k u_k \rangle \right) + \nabla \cdot \gamma_A \alpha_k \, {}^{2i}\langle \rho_k \underline{U}_k u_k \rangle$$

$$= - \gamma_v \alpha_k \, {}^{3i}\langle P_k \nabla \cdot \underline{U}_k \rangle - \nabla \cdot \gamma_A \alpha_k \, {}^{2i}\langle \underline{J}_{qk} \rangle$$

$$+ \gamma_v \alpha_k \left({}^{3i}\langle J_{Ek} \rangle + {}^{3i}\langle \phi_k \rangle + {}^{3i}\langle \dot{Q}_{wk} \rangle \right)$$

$$- v^{-1} \int_{A_{kf}} \underline{J}_{qk} \cdot \underline{n}_k \, dA - v^{-1} \int_{A_{kf}} \rho_k u_k (\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k \, dA. \qquad (3.45)$$

Again, we note

$$v^{-1} \int_{A_{kf}} \rho_k u_{kf} (\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k \, dA = v^{-1} \int_{A_{kf}} \rho_k (u_k - u_{kf})(\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k \, dA$$

$$- \gamma_v \Gamma_k u_s',$$

where

$$u_s' = \frac{\int_{A_{kf}} \rho_k u_{kf} (\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k \, dA}{\int_{A_{kf}} \rho_k (\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k \, dA}, \qquad (3.46)$$

which is the mean interface internal energy associated with generation of phase k over A_{kf}.

Also,

$$- v^{-1} \int_{A_{kf}} \underline{J}_{qk} \cdot \underline{n}_k \, dA = \gamma_v \, {}^{3m}\langle \dot{Q}_{kf} \rangle = \gamma_v \, \alpha_k \, {}^{3i}\langle \dot{Q}_{kf} \rangle, \qquad (3.47)$$

as already defined in Eq. 3.36.

The volume-averaged interfacial internal energy balance is

$$v^{-1} \int_{A_{kf}} \rho_k u_k (\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k \, dA + v^{-1} \int_{A_{kf}} \underline{J}_{qk} \cdot \underline{n}_k \, dA$$

$$+ v^{-1} \int_{A_{kf}} \rho_f u_f (\underline{U}_f - \underline{U}_{kf}) \cdot \underline{n}_f \, dA + v^{-1} \int_{A_{kf}} \underline{J}_{qf} \cdot \underline{n}_f \, dA = 0. \tag{3.48}$$

3.4 Volume Averaged Energy Equation in Terms of Enthalpy

For phase k, we have

$$\frac{\partial(\rho_k h_k)}{\partial t} + \nabla \cdot (\rho_k \underline{U}_k h_k) = \frac{dP_k}{dt} - \nabla \cdot \underline{J}_{qk} + J_{Ek} + \Phi_k, \tag{3.49}$$

which is valid for an ideal gas. For constant ρ_k, the dP_k/dt term should be deleted regardless of whether the flow is steady or unsteady. Otherwise,

$$\frac{dP_k}{dt} = \frac{\partial P_k}{\partial t} + \underline{U}_k \cdot \nabla P_k$$

$$= \frac{\partial P_k}{\partial t} + \nabla \cdot (P_k \underline{U}_k) - P_k \nabla \cdot \underline{U}_k. \tag{3.50}$$

The local volume average of Eq. 3.49 is

$$\gamma_v \frac{\partial}{\partial t} {}^{3m}\langle \rho_k h_k \rangle + \nabla \cdot \gamma_A {}^{2m}\langle \rho_k \underline{U}_k h_k \rangle$$

$$= \gamma_v \frac{\partial}{\partial t} {}^{3m}\langle P_k \rangle + \nabla \cdot \gamma_A {}^{2m}\langle P_k \underline{U}_k \rangle - \gamma_v {}^{3m}\langle P_k \nabla \cdot \underline{U}_k \rangle$$

$$- \nabla \cdot \gamma_A {}^{2m}\langle \underline{J}_{qk} \rangle + \gamma_v \left({}^{3m}\langle J_{Ek} \rangle + {}^{3m}\langle \Phi_k \rangle + {}^{3m}\langle \dot{Q}_{wk} \rangle \right)$$

$$+ v^{-1} \int_{A_{kf}} P_k (\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k \, dA - v^{-1} \int_{A_{kf}} \underline{J}_{qk} \cdot \underline{n}_k \, dA$$

$$- v^{-1} \int_{A_{kf}} \rho_k h_k (\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k \, dA. \tag{3.51}$$

In terms of intrinsic averages, Eq. 3.51 becomes

$$\gamma_v \frac{\partial}{\partial t} \left(\alpha_k {}^{3i}\langle \rho_k h_k \rangle \right) + \nabla \cdot \gamma_A \alpha_k {}^{2i}\langle \rho_k \underline{U}_k h_k \rangle$$

$$= \gamma_v \frac{\partial}{\partial t} \left(\alpha_k {}^{3i}\langle P_k \rangle \right) + \nabla \cdot \gamma_A \alpha_k {}^{2i}\langle P_k \underline{U}_k \rangle - \gamma_v \alpha_k {}^{3i}\langle P_k \nabla \cdot \underline{U}_k \rangle$$

$$- \nabla \cdot \gamma_A \alpha_k \, {}^{2i}\langle \underline{J}_{qk} \rangle + \gamma_v \alpha_k \left({}^{3i}\langle J_{Ek} \rangle + {}^{3i}\langle \phi_k \rangle + {}^{3i}\langle \dot{Q}_{wk} \rangle \right)$$

$$+ v^{-1} \int_{A_{kf}} P_k (\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k \, dA - v^{-1} \int_{A_{kf}} \underline{J}_{qk} \cdot \underline{n}_k \, dA$$

$$- v^{-1} \int_{A_{kf}} \rho_k h_k (\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k \, dA. \qquad (3.52)$$

We again note

$$v^{-1} \int_{A_{kf}} \rho_k h_k \, (\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k \, dA = v^{-1} \int_{A_{kf}} \rho_k (h_k - h_{kf})(\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k \, dA$$

$$- \gamma_v \Gamma_k \, h'_s, \qquad (3.53)$$

where

$$h'_s = \frac{\int_{A_{kf}} \rho_k h_{kf} (\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k \, dA}{\int_{A_{kf}} \rho_k (\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k \, dA}, \qquad (3.54)$$

which is the mean interface enthalpy associated with the generation of phase k over A_{kf}.

For later reference, we shall consider the case in which the intrinsic average of products is the same as the product of the intrinsic averages. We have then

$$\gamma_v \frac{\partial}{\partial t} \left(\alpha_k \, {}^{3i}\langle \rho_k \rangle \, {}^{3i}\langle h_k \rangle \right) + \nabla \cdot \gamma_A \alpha_k \, {}^{2i}\langle \rho_k \rangle \, {}^{2i}\langle \underline{U}_k \rangle \, {}^{2i}\langle h_k \rangle$$

$$= \gamma \frac{d}{dt} \left(\alpha_k \, {}^{3i}\langle P_k \rangle \right) - \nabla \cdot \gamma_A \alpha_k \, {}^{2i}\langle \underline{J}_{qk} \rangle$$

$$+ \gamma_v \alpha_k \left({}^{3i}\langle J_{Ek} \rangle + {}^{3i}\langle \phi_k \rangle + {}^{3i}\langle \dot{Q}_{wk} \rangle \right)$$

$$+ \gamma_v \alpha_k \, {}^{3i}\langle \dot{Q}_{kf} \rangle + \Gamma_k h_{kf} + v^{-1} \int_{A_{kf}} P_k \, (\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k \, dA, \qquad (3.55)$$

where

$$\frac{d}{dt} \, \alpha_k \, {}^{3i}\langle P_k \rangle = \frac{\partial}{\partial t} \, \alpha_k \, {}^{3i}\langle P_k \rangle + {}^{3i}\langle \underline{U}_k \rangle \cdot \nabla \, \alpha_k \, {}^{3i}\langle P_k \rangle$$

$$= \frac{\partial}{\partial t} \alpha_k \, {}^{3i}\langle P_k \rangle + \nabla \cdot \alpha_k \, {}^{3i}\langle P_k \rangle \, {}^{3i}\langle \underline{U}_k \rangle - \alpha_k \, {}^{3i}\langle P_k \rangle \nabla \cdot {}^{3i}\langle \underline{U}_k \rangle, \tag{3.56}$$

h_{kf} is defined by

$$- v^{-1} \int_{A_{kf}} \rho_k h_k (\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k \, dA = \gamma_v \, \Gamma_k \, h_{kf}, \tag{3.57}$$

and ${}^{3i}\langle \dot{Q}_{kf} \rangle$ is defined in Eq. 3.47.

A somewhat shorter form of Eq. 3.52 is

$$\begin{aligned} &\gamma_v \frac{\partial}{\partial t} \left(\alpha_k \, {}^{3i}\langle \rho_k h_k - P_k \rangle \right) + \nabla \cdot \gamma_A \alpha_k \, {}^{2i}\langle (\rho_k h_k - P_k) \, \underline{U}_k \rangle \\ &= - \gamma_v \alpha_k \, {}^{3i}\langle P_k \nabla \cdot \underline{U}_k \rangle - \nabla \cdot \gamma_A \alpha_k \, {}^{2i}\langle \underline{J}_{qk} \rangle \\ &+ \gamma_v \alpha_k \left({}^{3i}\langle \underline{J}_{Ek} \rangle + {}^{3i}\langle \Phi_k \rangle + {}^{3i}\langle \dot{Q}_{wk} \rangle + \gamma_v \, {}^{3m}\langle \dot{Q}_{kf} \rangle \right) \\ &- v^{-1} \int_{A_{kf}} (\rho_k h_k - P_k)(\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k \, dA \end{aligned} \tag{3.58}$$

and a similar form for Eq. 3.51.

When ρ_k = constant, Eq. 3.51 simplifies to

$$\begin{aligned} &\gamma_v \rho_k \frac{\partial}{\partial t} \, {}^{3m}\langle h_k \rangle + \rho_k \nabla \cdot \gamma_A \, {}^{2m}\langle \underline{U}_k h_k \rangle \\ &= - \nabla \cdot \gamma_A \, {}^{2m}\langle \underline{J}_{qk} \rangle + \gamma_v \left({}^{3m}\langle \underline{J}_{Ek} \rangle + {}^{3m}\langle \Phi_k \rangle + {}^{3m}\langle \dot{Q}_{wk} \rangle + {}^{3m}\langle \dot{Q}_{kf} \rangle \right) \\ &- v^{-1} \rho_k \int_{A_{kf}} h_k (\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k \, dA \end{aligned} \tag{3.59}$$

and a similar equation for the intrinsic averaged quantities.

4. INFLUENCE OF FLOW CONFIGURATIONS

Equations 3.10, 3.22, and 3.40, or Eqs. 3.11, 3.25, and 3.41 are, in essence, basic equations of a general flow system with interfaces. They are differential-integral equations with integrals based on probabilistic phase configuration of various moments in the nature of general transport equations (for instance, Kac, 1959). These integrals are

(a) For Mass Transfer

$$v^{-1} \int_{A_{kf}} \rho_k (\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k \, dA = - \gamma_v \Gamma_k$$

(b) For Momentum Transfer

$$v^{-1} \int_{A_{kf}} [\rho_k \underline{U}_k (\underline{U}_k - \underline{U}_{kf}) + P_k \underline{\underline{I}} - \underline{\underline{\tau}}_k] \cdot \underline{n}_k \, dA$$

(c) For Total Energy Transfer

$$v^{-1} \int_{A_{kf}} [\rho_k E_k (\underline{U}_k - \underline{U}_{kf}) + P_k \underline{U}_k - \underline{\underline{\tau}}_k \cdot \underline{U}_k + \underline{J}_{qk}] \cdot \underline{n}_k \, dA$$

and others involving internal energy or enthalpy. A common nature of these transport equations is the presence of integrals representing transfer of mass, momentum, and energy across the interface with variables $\underline{U}_{kf}$, $\underline{n}_k$, A_{kf} etc. These interface variables are determined by the interface balance relations of Eqs. 3.12, 3.29, and 3.42, while the interface geometry is dependent on the dynamic variables (Soo 1981).

Direct solution of the differential-integral equations 3.10, 3.22, and 3.40 (or Eqs. 3.11, 3.25, and 3.41) remains to be devised, and in short term some empirical approximation needs to be introduced (Sha and Soo, 1981). Other than the integral terms, however, the finite-difference formulation based on staggered grid for single-phase flow (Sha et al., 1978; Sha and Chao, 1981) is readily applied.

4.1 Pure Stratified Flow

Significant simplification is made possible by treating the system as an approximation of pure stratifed flows, that is, where transfer of inertia force and pressure force across the interface is negligible. Most of the intuitively adapted two-phase formulations follow this route. Accordingly, the momentum-transfer integral defined in Eq. 3.20 reduces to

$$v^{-1} \int_{A_{kf}} \rho_k \underline{U}_k (\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k \, dA = - \gamma_v \Gamma_k \underline{U}'_s . \qquad (4.1)$$

since, by definition, a pure stratifed flow is one having plane interfaces and is best represented by constant interface velocity and pressure within the control volume. Then

$$v^{-1} \int_{A_{kf}} \rho_k (\underline{U}_k - \underline{U}_{kf})(\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k \, dA = 0. \tag{4.2}$$

We define

$$v^{-1} \int_{A_{kf}} \underline{\underline{\tau}}_k \cdot \underline{n}_k \, dA = \gamma_v \underline{V}_{kf}, \tag{4.3}$$

where $\underline{V}_{kf}$ is the interfacial viscous drag force per unit volume of the mixture. The energy-transfer integral also reduces according to

$$- v^{-1} \int_{A_{kf}} [\rho_k E_k (\underline{U}_k - \underline{U}_{kf}) + P_k \underline{U}_k - \underline{\underline{\tau}}_k \cdot \underline{U}_k + \underline{J}_{qk}] \cdot \underline{n}_k \, dA$$

$$= \gamma_v \left(\Gamma_k E'_s + P \, \underline{U}'_s \cdot \nabla \alpha_k + \underline{U}'_s \cdot \underline{V}_{kf} \right) + \gamma_v \, {}^{3m}\langle \delta_{kf} \rangle, \tag{4.4}$$

where $\underline{U}'_s$ is the constant interface velocity.

Similar reduction is applicable to the transfer integral in terms of internal energy or enthalpy. A case of such an application is that the TRAC code (Liles etal., 1979) can be modified to account for flow through structured regions. Equations 3.11, 3.25, and 3.44 (with Eqs. 3.56, 4.2, and 4.3) become, with the assumption of averages of products being equal to products of averages,

$$\gamma_v (\partial \alpha_k \bar{\rho}_k / \partial t) + \nabla \cdot (\gamma_A \alpha_k \bar{\rho}_k \underline{U}_k) = \gamma_v \Gamma_k \tag{4.5}$$

and

$$\gamma_v (\partial \alpha_k \bar{\rho}_k \underline{U}_k / \partial t) + \nabla \cdot (\gamma_A \alpha_k \bar{\rho}_k \underline{U}_k \underline{U}_k)$$

$$= - \gamma_v \alpha_k \nabla P + \nabla \cdot \gamma_A \underline{\underline{\tau}}_{mk} + \gamma_v (\underline{V}_{kf} + \Gamma_k \underline{U}_m + \underline{f} \, \alpha_k \bar{\rho}_k), \tag{4.6}$$

in which

$P = {}^{3i}\langle P \rangle$, $\underline{\underline{\tau}}_{mk} = {}^{2i}\langle \underline{\underline{\tau}}_k \rangle$, $\underline{U}_m = \underline{U}_s$, and $- v^{-1} \int_{A_{kf}} P \cdot \underline{n}_k \, dA = \gamma_v P \nabla \alpha_k$, according to the specification of stratified flow. The latter corresponds to the case of $B_k = 1$ in the representation of Sha and Soo (1979). And,

$$\gamma_v (\partial \alpha_k \bar{\rho}_k u_k / \partial t) + \nabla \cdot (\gamma_A \alpha_k \bar{\rho}_k \underline{U}_k u_k)$$

$$= - P \nabla \cdot \gamma_A \alpha_k \underline{U}_k - P \gamma_v (\partial \alpha_k / \partial t) - \nabla \cdot \gamma_A \underline{J}_{qmk}$$

$$- \gamma_v J_{Ek} + \gamma_v \dot{Q}_{kf} + \gamma_v \Gamma_k h_{kf}, \qquad (4.7)$$

in which $\underline{J}_{qmk} = {}^{2m}\langle \underline{J}_{qk} \rangle$, $J_{Ek} = {}^{3m}\langle J_{Ek} \rangle$, and $\dot{Q}_{kf} = {}^{3m}\langle \dot{Q}_{kf} \rangle$. The dissipation term is neglected in Eq. 4.7, and $\underline{\tau}_{mk}$ and $\underline{J}_{qmk}$ are viscous stresses and heat flux of phase k in the mixture; h_{kf} (< 0, when k is the bubble phase) is the latent heat of phase change from k to f at pressure P of the mixture. The mixture equations corresponding to the TRAC code are given in Sec. 4 of the appendix.

4.2 Highly Dispersed Mixture

The meaning of the interface transfer integrals in the averaged phasic equations is further seen from the equations of the mixtures. Note that Eqs. A.14 and A.16 in the Appendix reduce to that of a simple molecular mixture when

$$v^{-1} \int_{A_{kf}} \rho_k (\underline{U}_k - \underline{U}_{kf})(\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k \, dA$$

$$= - \nabla \cdot \gamma_A \alpha_k \, {}^{2i}\langle \rho_k (\underline{U}_k - \underline{U}'_s)(\underline{U}_k - \underline{U}_m) \rangle \qquad (4.8)$$

and

$$v^{-1} \int_{A_{kf}} \rho_k (E_k - E_{kf})(\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k \, dA$$

$$= - \nabla \cdot \gamma_A \alpha_k \, {}^{2i}\langle \rho_k (E_k - E'_s)(\underline{U}_k - \underline{U}_m) \rangle \qquad (4.9)$$

suggesting that these equalities represent a limiting condition when phase k is dispersed throughout the mixture.

When an approximation of a homogeneous mixture with disparate phase velocities and temperatures can be made,

$$- v^{-1} \int_{A_{kf}} [\rho_k \underline{U}_k (\underline{U}_k - \underline{U}_{kf}) + P_k \underline{\underline{I}} - \underline{\tau}_k] \cdot \underline{n}_k \, dA$$

$$= - \nabla \cdot \gamma_A \, {}^{2m}\langle \rho_k (\underline{U}_k - \underline{U}_m)(\underline{U}_k - \underline{U}'_s) \rangle + \gamma_v \Gamma_k \underline{U}'_s + \gamma_v \underline{V}_{ks}, \qquad (4.10)$$

where $\gamma_v \underline{V}_{ks} = v^{-1} \int_{A_{kf}} (-P \underline{\underline{I}} + \underline{\tau}_k) \cdot n_k \, dA$, and

$$- v^{-1} \int_{A_{kf}} [\rho_k (E_k - E_{kf})(\underline{U}_k - \underline{U}_{kf}) + P_k \underline{U}_k - \underline{\tau}_k \cdot \underline{U}_k + \underline{J}_{qk}] \cdot \underline{n}_k \, dA$$

$$= \nabla \cdot \gamma_A \, {}^{2m}\langle \rho_k (E_k - E'_s)(\underline{U}_k - \underline{U}_m) \rangle + \gamma_v \, {}^{3m}\langle \underline{U}_{ks} \cdot \underline{V}_{ks} \rangle + \gamma_v \, {}^{3m}\langle \dot{Q}_{kf} \rangle. \quad (4.11)$$

4.3 Other Limiting Conditions

We note that in the absence of motion, the hydrostatic relation is satisfied, according to Eq. 3.26,

$$0 = - \nabla \, {}^{3i}\langle P_k \rangle + {}^{3i}\langle \rho_k \rangle \, \underline{f}, \quad (4.12)$$

and Eq. A.14

$$0 = - \nabla \, {}^{3i}\langle P_m \rangle + {}^{3i}\langle \rho_m \rangle \, \underline{f} \quad (4.13)$$

as expected.

The Bernoulli relation for an incompressible fluid, when applied to the present system, however, takes the form of

$$\frac{1}{2} \, {}^{3i}\langle \rho_k \rangle \, \gamma_v^{-1} \, (\gamma_{Ax} U_{kx}^2 + \gamma_{Ay} U_{ky}^2 + \gamma_{Az} U_{kz}^2) + {}^{3i}\langle P_k \rangle$$

$$- \, {}^{3i}\langle \rho_k \rangle \, \underline{g} = \text{constant}, \quad (4.14)$$

and for the mixture (Appendix),

$$\frac{1}{2} \, \rho_m \, \gamma_v^{-1} \, (\gamma_{Ax} U_{mx}^2 + \gamma_{Ay} U_{my}^2 + \gamma_{Az} U_{mz}^2) + {}^{3i}\langle P \rangle$$

$$- \, {}^{3i}\langle \rho_m \rangle \, \underline{g} + \frac{1}{2} \sum_k \alpha_k \, {}^{3i}\langle \rho_k \rangle \, \gamma_v^{-1} \, [\gamma_{Ax} (U_{kx} - U_{mx})^2$$

$$+ \gamma_{Ay} (U_{ky} - U_{my})^2 + \gamma_{Az} (U_{kz} - U_{mz})^2], \quad (4.15)$$

when applied to pure stratified flow.

For an ideal dispersed mixture, however, we have

$$\frac{1}{2} \, {}^{3i}\langle \rho_k \rangle \, \gamma_v^{-1} \, [\gamma_{Ax} U_{kx}^2 - \gamma_{Ax} (U_{kx} - U_{mx})^2$$

$$+ \gamma_{Ay} U_{ky}^2 - \gamma_{Ay} (U_{ky} - U_{my})^2 + \gamma_{Az} U_{kz}^2 - \gamma_{Az} (U_{kz} - U_{mz})^2]$$

$$+ {}^{3i}\langle P_k \rangle - {}^{3i}\langle \rho_k \rangle \underline{g} = \text{constant} \tag{4.16}$$

for phase k, and for the mixture, we have

$$\frac{1}{2} {}^{3i}\langle \rho_m \rangle \gamma_v^{-1} \left(\gamma_{Ax} U_{mx}^2 + \gamma_{Ay} U_{my}^2 + \gamma_{Az} U_{mz}^2 \right) + {}^{3i}\langle P \rangle$$

$$- {}^{3i}\langle \rho_m \rangle = \text{constant.} \tag{4.17}$$

The corresponding relations for the general flow configurations are expected to fall within these two limiting cases.

5. FORMULATION FOR COMMIX-2

Simplifications of the general differential-integral equations lead to the governing equations used in COMMIX-2 code (Domanus et al., 1980) for the two-phase computations at the present stage of development. They have been made possible via the following specifications and assumptions:

(1) The flow configuration is such that interaction between phases can be represented by drag force and convection of heat only, and the following integrals hold:

$$v^{-1} \int_{A_{kf}} \underline{P}_k \, n \, dA = - \gamma_v \, {}^{3i}\langle P_k \rangle \nabla \alpha_k$$

and

$$\int_{A_{kf}} P_k \, (\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k \, dA = 0.$$

The effect of wave propagation is assumed to be negligible.

(2) The intrinsic averages of products are equal to products of the intrinsic averages.

(3) The drag force between the phases is represented by

$$v^{-1} \int_{A_{kf}} \underline{\tau}_k \cdot \underline{n}_k \, dA \equiv \gamma_v \underline{V}_{kf} = \gamma_v K (\underline{U}_f - \underline{U}_k) \tag{5.1}$$

for the interface force acting on phase k by phase f. As an example, the friction function K may be given by

$$K \cong \frac{3}{4} \alpha_k \overline{\rho}_f C_D |\underline{U}_f - \underline{U}_k| (D_h)^{-1}, \tag{5.2}$$

where C_D is the drag coefficient for flow of f over k, and D_h is the characteristic hydraulic diameter of phase k (Harlow and Asmden, 1975).

(4) The heat-transfer coupling for convective heat transfer from phase f to phase k is represented by

$$- v^{-1} \int_{A_{kf}} \underline{J}_{qk} \cdot \underline{n}_k \, dA \equiv \gamma_v \mathring{Q}_{kf} \triangleq \gamma_v R(T_f - T_k). \tag{5.3}$$

As an example, the heat-transfer function R may be given by

$$R \cong 6 \alpha_k (Nu) \kappa_f / (D_h)^2, \tag{5.4}$$

where Nu is the Nusselt number of convection for flow of f over particles k, κ_f is the thermal conductivity of fluid f, and c_{pk} is the specific heat of phase k.

(5) The shear stress acting on phase k is assumed to be given by $\alpha_k \, {}^{3i}\langle \underline{\tau}_k \rangle \equiv \alpha_k \, \overline{\underline{\tau}}_k$, and the heat flux through phase k is assumed to be given by $\underline{J}_{qk} = - \alpha_k \, \kappa_k \, \nabla \, T_k$. These relations also conform to the stratified flow model. The ramification of general approximations are discussed in Soo (1967) and in Sha and Soo (1980). Other details are given in Domanus et al. (1980).

The basic equations for phase k now take the following forms:

From Eq. 3.11,

$$\gamma_v \left(\partial \alpha_k \overline{\rho}_k / \partial t \right) + \frac{\partial}{\partial x} \gamma_{Ax} \, \alpha_k \, \overline{\rho}_k U_{kx} + \frac{\partial}{\partial y} \gamma_{Ay} \, \alpha_k \, \overline{\rho}_k \, U_{ky} + \frac{\partial}{\partial z} \gamma_{Az} \, \alpha_k \, \overline{\rho}_k \, U_{kz} = \gamma_v \, \Gamma_k . \tag{5.5}$$

From Eq. 3.25

$$\begin{aligned} & \gamma_v \left(\partial \alpha_k \overline{\rho}_k U_{kx} / \partial t \right) + \frac{\partial}{\partial x} \gamma_{Ax} \, \alpha_k \, \overline{\rho}_k \, U_{kx}^2 \\ & + \frac{\partial}{\partial y} \gamma_{Ay} \, \alpha_k \, \overline{\rho}_k \, U_{kx} \, U_{ky} + \frac{\partial}{\partial z} \gamma_{Az} \, \alpha_k \, \overline{\rho}_k \, U_{kx} \, U_{kz} \\ & = \gamma_v \, \alpha_k \, \overline{\rho}_k \, g_x - \gamma_v \, \alpha_k \, \partial P / \partial x + \frac{\partial}{\partial z} \gamma_{Ax} \, \alpha_k \, \overline{\tau}_{kxx} \\ & + \frac{\partial}{\partial y} \gamma_{Ay} \, \alpha_k \, \overline{\tau}_{kyx} + \frac{\partial}{\partial z} \gamma_{Az} \, \alpha_k \, \overline{\tau}_{kzx} + \gamma_v \, K \left(U_{fx} - U_{kx} \right) \\ & - \gamma_v \, R_x + \gamma_v \, \Gamma_k \, U_{mx}, \end{aligned} \tag{5.6}$$

in which $R_x = {}^{3m}\langle R_x \rangle$. The y and z components of the momentum equation can be readily written down.

From Eq. 3.55,

$$\begin{aligned} & \gamma_v \left(\partial \alpha_k \overline{\rho}_k h_k / \partial t \right) + \frac{\partial}{\partial x} \gamma_{Ax} \, \alpha_k \, \overline{\rho}_k \, U_{kx} \, h_k \\ & + \frac{\partial}{\partial y} \gamma_{Ay} \, \alpha_k \, \overline{\rho}_k \, U_{ky} \, h_k + \frac{\partial}{\partial z} \gamma_{Az} \, \alpha_k \, \overline{\rho}_k \, U_{kz} \, h_k \\ & = \gamma_v \left(d \alpha_k P / dt \right) + \gamma_v \left[\dot{Q}_{wk} + \dot{Q}_{kf} + \Phi_k + R \left(T_f - T_k \right) + \Gamma_k h_{kf} \right] \end{aligned}$$

$$+ \frac{\partial}{\partial x} \gamma_{Ax} \alpha_k \kappa_k \frac{\partial T_k}{\partial x} + \frac{\partial}{\partial y} \gamma_{Ay} \alpha_k \kappa_k \frac{\partial T_k}{\partial y} + \frac{\partial}{\partial z} \gamma_{Az} \alpha_k \kappa_k \frac{\partial T_k}{\partial z}, \quad (5.7)$$

in which $\dot{Q}_{wk} = {}^{3m}\langle \dot{Q}_{wk} \rangle$, $\dot{Q}_{kf} = {}^{3m}\langle \dot{Q}_{kf} \rangle$, and $\Phi_k = {}^{3m}\langle \Phi_k \rangle$.

These equations may be further refined to closely approximate the basic differential-integral equations. An alternate route is to perform experimental validation to determine the limit of application of the code based on Eqs. 5.5-5.7.

6. CONCLUDING REMARKS

A rigorous procedure will call for solving the differential-integral equations 3.10, 3.22 and 3.40 in the long run. In short terms, empirical interpolation between the limiting cases discussed in Chapter 4 will be useful (Sha and Soo, 1981). The evaluation of these probability integrals will require information on the interface area (Ishii and Mishima, 1980) and configuration of phases (Sha and Soo 1981, Soo 1981).

The present procedure extends from Sha and Chao (1981) to include the multiphase fluid. The use of volume porosity and nonisotropic surface permeabilities differs from the formulations based on mean interstitial flow area of Slattery (1967), Whitaker (1969), and Grey (1975). Since the velocity components are based on these surface permeabilities, there is no direct correspondence to the distributed mean velocities of these authors. Physically, the velocities from our formulation are based on surface flow area in a given direction. The present procedure has an advantage over those formulations based on mean interstitial openings in that baffles and blockages can be accounted for in a rational manner.

The restriction of

(Characteristic length of pores and phases)

$\ll$ (Characteristic length of averaging volume)

$\ll$ (Characteristic dimension of the physical system)

remains for the present method as with other porous media formulations.

Equations 5.5-5.7, on which the current code of COMMIX-2 is based, are seen to be, within the above stated degree of approximation, reasonable with the present knowledge of understanding of multiphase flow, although improvements seem feasible.

In summary, a set of governing equations for a multiphase system based on local volume average has been derived and presented. This set of governing equations is particularly suitable for numerical analysis for an isotropic medium. Approximations used in the present two-phase and multiphase flow have been delineated. In order to provide fundamental understanding of multiphase systems, future research to refine and possibly eliminate these approximations based on physical evidence is urgently needed.

APPENDIX

MIXTURE RELATIONS

The significance of the relations of phasic quantities to mixture quantities and the configuration of flow system has been illustrated in Sec. 1.2. Definitive limiting cases of pure stratified flow and highly dispersed flow are further seen in Secs. 4.1 and 4.2. Conservation relations as applied to the mixture of phases provide for additional understanding of interactions among phases. One notes that summing of phasic equations does not give rise to additional independent equations.

1. Volume Averaging as Applied to the Mixture of Phases

The conservation principles must also apply to an overall multiphase mixture. Summing over the phases, we get, for the mixture (subscript m), according to Eqs. 2.9, 2.10, 2.12, and 2.22,

$$ {}^{3m}\langle \rho_m \rangle = \sum_k {}^{3m}\langle \rho_k \rangle = {}^{3i}\langle \rho_m \rangle = \sum_k \alpha_k \, {}^{3i}\langle \rho_k \rangle, \tag{A.1} $$

$$ {}^{2m}\langle \rho_m \underline{U}_m \rangle = \sum_k {}^{2m}\langle \rho_k \underline{U}_k \rangle = {}^{2i}\langle \rho_m \underline{U}_m \rangle = \sum_k \alpha_k \, {}^{2i}\langle \rho_k U_k \rangle, \tag{A.2} $$

and

$$ {}^{3m}\langle \rho_m E_m \rangle = \sum_k {}^{3m}\langle \rho_k E_k \rangle = {}^{3i}\langle \rho_m E_m \rangle. \tag{A.3} $$

Equations A.1 and A.2 now give (Soo, 1967)

$$ {}^{2m}\langle \rho_m \underline{U}_m \underline{U}_m \rangle = \sum_k \left[{}^{2m}\langle \rho_k \underline{U}_k \underline{U}_k \rangle - {}^{2m}\langle \rho_k (\underline{U}_k - \underline{U}'_s)(\underline{U}_k - \underline{U}_m) \rangle \right] $$

$$ = {}^{2i}\langle \rho_m \underline{U}_m \underline{U}_m \rangle, \tag{A.4} $$

where $\underline{U}'_s$ is a velocity given by Eq. 3.21, a variable, or a constant; it is specific when determined from the interaction of phases. Equations A.1-A.3 further give

$$ {}^{2m}\langle \rho_m \underline{U}_m E_m \rangle = \sum_k \left[{}^{2m}\langle \rho_k \underline{U}_k E_k \rangle - {}^{2m}\langle \rho_k (\underline{U}_k - \underline{U}_m)(E_k - E'_s) \rangle \right] $$

$$ = {}^{2i}\langle \rho_m \underline{U}_m E_m \rangle. \tag{A.5} $$

Similar to $\underline{U}_s'$, E_s' again is an energy given by Eq. 3.39, a variable, or a constant. The definition of E and Eq. A.3 give

$$
\begin{aligned}
{}^{3m}\langle \rho_m \underline{u}_m \rangle &= \sum_k \left[{}^{3m}\langle \rho_k \underline{u}_k \rangle + (1/2)\, {}^{3m}\langle \rho_k (\underline{U}_k - \underline{U}_m) \cdot (\underline{U}_k - \underline{U}_m) \rangle \right] \\
&= {}^{3i}\langle \rho_m \underline{u}_m \rangle . \qquad \text{(A.6)}
\end{aligned}
$$

Neglecting capillary effect, we have

$$
{}^{3m}\langle P_m \rangle = \sum_k {}^{3m}\langle P_k \rangle = {}^{3i}\langle P_m \rangle = P = \sum_k \alpha_k \, {}^{3i}\langle P_k \rangle , \qquad \text{(A.7)}
$$

where ${}^{3m}\langle P_k \rangle$ corresponds to a "partial pressure,"

$$
{}^{2m}\langle \underline{U}_m P_m \rangle = \sum_k \left[{}^{2m}\langle \underline{U}_k P_k \rangle + {}^{2m}\langle (\underline{U}_m - \underline{U}_k) P_k \rangle \right], \qquad \text{(A.8)}
$$

and

$$
{}^{2m}\langle \underline{\tau}_m \rangle = \sum_k {}^{2m}\langle \underline{\tau}_k \rangle = {}^{2i}\langle \underline{\tau}_m \rangle = \sum_k \alpha_k \, {}^{2i}\langle \underline{\tau}_k \rangle . \qquad \text{(A.9)}
$$

Shear stresses in terms of ${}^{2m}\langle \underline{\tau}_k \rangle$ include transport of momenta via bodily displaced phases. An instance is the gradient of momentum of a dispersed phase arising from diffusion. Shear stress in terms of ${}^{2i}\langle \underline{\tau}_k \rangle$ is pertinent to stratified flow, however, and

$$
\begin{aligned}
{}^{2m}\langle \underline{U}_m \cdot \underline{\tau}_m \rangle &= \sum_k \left[{}^{2m}\langle \underline{U}_k \cdot \underline{\tau}_k \rangle + {}^{2m}\langle (\underline{U}_m - \underline{U}_k) \cdot \underline{\tau}_k \rangle \right] \\
&= {}^{2i}\langle \underline{U}_m \cdot \underline{\tau}_m \rangle . \qquad \text{(A.10)}
\end{aligned}
$$

The last equalities are according to the definitions of the shear stress of component k in the mixture.

The resistance force of the solid body is given by

$$
\sum_k {}^{3m}\langle R_k \rangle = {}^{3i}\langle \underline{R}_m \rangle . \qquad \text{(A.11)}
$$

The continuity equation (Eq. 3.11) is readily summed over all the phases to give

$$\gamma_v\left(\partial\,{}^{3i}\langle\rho_m\rangle/\partial t\right) + \nabla\,\gamma_A\,{}^{2i}\langle\rho_{m\underline{m}}\underline{U}_{\underline{m}}\rangle = 0, \tag{A.12}$$

since conservation of mass at the interface gives

$$\sum_k \int_{A_k} \rho_k\left(\underline{U}_k - \underline{U}_{kf}\right)\cdot\underline{n}_k\,dA = 0. \tag{A.13}$$

Summing over the linear momentum equations of all the phases and using Eq. A.4 yield (note that γ_A and γ_v are independent of k)

$$\gamma_v\frac{\partial}{\partial t}\,{}^{3i}\langle\rho_{m\underline{m}}\underline{U}_{\underline{m}}\rangle + \nabla\cdot\gamma_A\,{}^{2i}\langle\rho_{m\underline{m}}\underline{U}_{\underline{m}}\underline{U}_{\underline{m}}\rangle + v^{-1}\sum_k\int_{A_{kf}}\rho_k\,\underline{U}_k\left(\underline{U}_k - \underline{U}_s\right)\cdot\underline{n}_k\,dA$$

$$= -\gamma_v\,\nabla\,{}^{3i}\langle P_m\rangle + \nabla\cdot\gamma_A\,{}^{2i}\langle\underline{\tau}_{\underline{m}}\rangle + \gamma_v\,{}^{3i}\langle\rho_m\rangle\,\underline{f}$$

$$-\gamma_v\,{}^{3i}\langle\underline{R}_{\underline{m}}\rangle - \nabla\cdot\gamma_A\sum_k\,{}^{2m}\langle\rho_k\underline{U}_k\left(\underline{U}_k - \underline{U}_{\underline{m}}\right)\rangle. \tag{A.14}$$

Since the internal action and reaction of drag and diffusive forces cancel out among the phases,

$$\sum_k\int_{A_{kf}}\left(-P_k\underline{n}_k + \underline{\tau}_k\cdot\underline{n}_k\right)dA = 0. \tag{A.15}$$

Summing up over the energy equations for all k's and using Eq. A.5 yield

$$\gamma_v\frac{\partial}{\partial t}\,{}^{3i}\langle\rho_{m\underline{m}}\underline{E}_{\underline{m}}\rangle + \nabla\cdot\gamma_A\,{}^{2i}\langle\rho_{m\underline{m}}\underline{U}_{\underline{m}}\underline{E}_m\rangle + v^{-1}\sum_k\int_{A_{kf}}\rho_k\,\underline{E}_k\left(\underline{U}_k - \underline{U}_s\right)\cdot\underline{n}_k dA$$

$$= \nabla\cdot\gamma_A\,{}^{2i}\langle\left(-P_{m\underline{m}}\underline{I} + \underline{\tau}_{\underline{m}}\right)\cdot\underline{U}_{\underline{m}}\rangle - \nabla\cdot\gamma_A\,{}^{2i}\langle\underline{J}_{qm}\rangle$$

$$+\gamma_v\,{}^{3i}\langle\dot{Q}_{wm} + J_{Em}\rangle + \gamma_v\,{}^{3i}\langle\rho_{m\underline{m}}\underline{U}_{\underline{m}}\rangle\cdot\underline{f}$$

$$-\nabla\cdot\gamma_A\sum_k\,{}^{2m}\langle\left(-P_{k\underline{m}}\underline{I} + \underline{\tau}_{\underline{k}}\right)\cdot\left(\underline{U}_k - \underline{U}_{\underline{m}}\right)\rangle$$

$$-\nabla\cdot\gamma_A\sum_k\,{}^{2m}\langle\rho_k E_k\left(\underline{U}_k - \underline{U}_{\underline{m}}\right)\rangle. \tag{A.16}$$

Since $\sum_k \rho_k\left(\underline{U}_k - \underline{U}_{\underline{m}}\right) = 0$,

$$\sum_k\,{}^{2m}\langle\rho_k E_k\left(\underline{U}_k - \underline{U}_{\underline{m}}\right)\rangle = \sum_k\,{}^{2m}\langle\rho_k\left(E_k - E'_s\right)\left(\underline{U}_k - \underline{U}_{\underline{m}}\right)\rangle, \tag{A.17}$$

where E_s' is arbitrary, constant, or variable, except that it must be independent of k.

Further, the total heat transfer from the wall is

$$\sum_k \dot{Q}_{kw} = {}^{3i}\langle \dot{Q}_w \rangle \tag{A.18}$$

and the mutual heat transfer between phases cancels out to give

$$\sum_k \int_{A_{kf}} \underline{J}_{qk} \cdot \underline{n}_k \, dA = 0. \tag{A.19}$$

The sum of the work done by one phase on the other also cancels out, so that

$$\sum_{k,f} \int_{A_{kf}} P_k \underline{U}_k \cdot \underline{n}_k \, dA = 0. \tag{A.20}$$

The dissipation in the system is given by

$$\sum_{k,f} \int_{A_{kf}} \underline{U}_k \cdot \underline{\tau}_k \cdot \underline{n}_k \, dA. \tag{A.21}$$

We note that the nature of the conservation relations is strongly influenced by the interface transfer integrals, and solution of these equations calls for expressing averages of products in terms of products of averages, taking proper account of their differences.

2. Energy Equation in Terms of Internal Energy

Alternatively, we may treat the mixtures on the basis of internal energy

$$\begin{aligned} {}^{3i}\langle \rho_m u_m \rangle &\equiv {}^{3m}\langle \rho_m u_m \rangle = \sum_k {}^{3m}\langle \rho_k u_k \rangle \\ &= \sum_k \alpha_k \, {}^{3i}\langle \rho_k u_k \rangle. \end{aligned} \tag{A.22}$$

A defining relationship for mixture internal energy u_m is $\rho_m u_m \equiv \sum_k \rho_k u_k$. Note that this will replace $\sum_k {}^{3m}\langle \rho_k E_k \rangle = {}^{3i}\langle \rho_m E_m \rangle$. Therefore,

$$\begin{aligned} {}^{2i}\langle \rho_m \underline{U}_m u_m \rangle &\equiv {}^{2m}\langle \rho_m \underline{U}_m u_m \rangle \\ &= \sum_k \left[{}^{2m}\langle \rho_k \underline{U}_k u_k \rangle - {}^{2m}\langle \rho_k u_k (\underline{U}_k - \underline{U}_m) \rangle \right] \end{aligned}$$

$$= \sum_k \left[{}^{2m}\langle \rho_k \underline{U}_k u_k \rangle - {}^{2m}\langle \rho_k (u_k - u_s')(\underline{U}_k - \underline{U}_m) \rangle \right], \qquad \text{(A.23)}$$

since $\sum_k \rho_k (\underline{U}_k - \underline{U}_m) = 0$, and u_s' is arbitrary except that it must be related to Eq. 3.46 independent of k. Equation A.23 can also be written as

$$\begin{aligned} {}^{2i}\langle \rho_m \underline{U}_m u_m \rangle &= \sum_k \alpha_k \left[{}^{2i}\langle \rho_k \underline{U}_k u_k \rangle - {}^{2i}\langle \rho_k u_k (\underline{U}_k - \underline{U}_m) \rangle \right] \\ &= \sum_k \alpha_k \left[{}^{2i}\langle \rho_k \underline{U}_k u_k \rangle - {}^{2i}\langle \rho_k (u_k - u_s')(\underline{U}_k - \underline{U}_m) \rangle \right] \qquad \text{(A.24)} \end{aligned}$$

We also note that $\sum_k \rho_k u_k (\underline{U}_k - \underline{U}_m) = \sum_k \rho_k \underline{U}_k (u_k - u_m)$, and hence other forms of Eq. A.23 are possible:

$$\begin{aligned} {}^{3i}\langle P_m \nabla \cdot \underline{U}_m \rangle &\equiv {}^{3m}\langle P_m \nabla \cdot \underline{U}_m \rangle \\ &\equiv \sum_k \left[{}^{3m}\langle P_k \nabla \cdot \underline{U}_k \rangle - {}^{3m}\langle P_k \nabla \cdot \underline{U}_k - P_m \nabla \cdot \underline{U}_m \rangle \right] \\ &= \sum_k \alpha_k \left[{}^{3i}\langle P_k \nabla \cdot \underline{U}_k \rangle - {}^{3i}\langle P_k \nabla \cdot (\underline{U}_k - \underline{U}_m) \rangle \right], \qquad \text{(A.25)} \end{aligned}$$

since $\sum_k P_k \nabla \cdot \underline{U}_m = \nabla \cdot \underline{U}_m \sum_k P_k = P_m \nabla \cdot \underline{U}_m$,

$${}^{2i}\langle \underline{J}_{qm} \rangle = {}^{2m}\langle \underline{J}_{qm} \rangle = \sum_k {}^{2m}\langle \underline{J}_{qk} \rangle = \sum_k \alpha_k \, {}^{2i}\langle \underline{J}_{qk} \rangle,$$

and

$${}^{3i}\langle \underline{J}_{qm} \rangle = {}^{3m}\langle \underline{J}_{qm} \rangle = \sum_k {}^{3m}\langle \underline{J}_{qk} \rangle = \sum_k \alpha_k \, {}^{3i}\langle \underline{J}_{qk} \rangle,$$

since $\underline{J}_{qm} \equiv \sum_k \underline{J}_{qk}$, a defining equation. Likewise,

$${}^{3i}\langle J_{Em} \rangle = {}^{3m}\langle J_{Em} \rangle = \sum_k \alpha_k \, {}^{3i}\langle J_{Ek} \rangle \,, \quad J_{Em} \equiv \sum_k J_{Ek};$$

$${}^{3i}\langle \phi_m \rangle = \sum_k \alpha_k \, {}^{3i}\langle \phi_k \rangle, \qquad \phi_m \equiv \sum_k \phi_k;$$

$$ {}^{3i}\langle \dot{Q}_{wm} \rangle = \sum_k \alpha_k \, {}^{3i}\langle \dot{Q}_{wk} \rangle \qquad\qquad \dot{Q}_{wm} \equiv \sum_k \dot{Q}_{wk}. $$

Summing up Eq. 3.45 for all k's and using Eq. 3.48 give

$$ \gamma_v \frac{\partial}{\partial t} {}^{3i}\langle \rho_m u_m \rangle + \nabla \cdot \gamma_A \, {}^{2i}\langle \rho_{m\text{-}m} \underline{U}_{m\text{-}m} u_m \rangle + v^{-1} \sum_k \int_{A_{kf}} \rho_k u_k (\underline{U}_k - \underline{U}_s) \cdot \underline{n}_k \, dA $$

$$ = - \gamma_v \, {}^{3i}\langle P_m \nabla \cdot \underline{U}_m \rangle - \nabla \cdot \gamma_A \, {}^{2i}\langle \underline{J}_{qm} \rangle + \gamma_v \, {}^{3i}\langle J_{\Sigma m} + \Phi_m + \dot{Q}_{wm} \rangle $$

$$ - \nabla \cdot \gamma_A \sum_k \alpha_k \, {}^{2i}\langle \rho_k u_k (\underline{U}_k - \underline{U}_m) \rangle $$

$$ - \gamma_v \sum_k \alpha_k \, {}^{3i}\langle P_k \nabla \cdot (\underline{U}_k - \underline{U}_m) \rangle. \qquad \text{(A.26)} $$

3. Energy Equation in Terms of Enthalpy

The local-volume averaged enthalpy equation for the mixture may be treated in a similar manner. We define

$$ {}^{3i}\langle \rho_m h_m \rangle \equiv {}^{3m}\langle \rho_m h_m \rangle = \sum_k {}^{3m}\langle \rho_k h_k \rangle $$

$$ = \sum_k \alpha_k \, {}^{3i}\langle \rho_k h_k \rangle. \qquad \text{(A.27)} $$

The relation $\rho_m h_m = \sum_k \rho_k h_k$ now replaces $\rho_m u_m = \sum_k \rho_k u_k$ and $\rho_m E_m = \sum_k \rho_k E_k$.

Accordingly,

$$ {}^{2i}\langle \rho_m h_{m\text{-}m} \underline{U}_{m\text{-}m} \rangle \equiv {}^{2m}\langle \rho_m h_{m\text{-}m} \underline{U}_{m\text{-}m} \rangle $$

$$ = \sum_k [{}^{2m}\langle \rho_k h_{k\text{-}k} \underline{U}_{k\text{-}k} \rangle - {}^{2m}\langle \rho_k h_k (\underline{U}_k - \underline{U}_m) \rangle] $$

$$ = \sum_k [{}^{2m}\langle \rho_k h_{k\text{-}k} \underline{U}_{k\text{-}k} \rangle - {}^{2m}\langle \rho_k (h_k - h'_s)(\underline{U}_k - \underline{U}_m) \rangle], \qquad \text{(A.28)} $$

where h'_s is arbitrary except that it must be related to Eq. 3.54. Or,

$$ {}^{2i}\langle \rho_m h_{m\text{-}m} \underline{U}_{m\text{-}m} \rangle = \sum_k [\alpha_k \, {}^{2i}\langle \rho_k h_{k\text{-}k} \underline{U}_{k\text{-}k} \rangle - \alpha_k \, {}^{2i}\langle \rho_k h_k (\underline{U}_k - \underline{U}_m) \rangle]. \qquad \text{(A.29)} $$

The enthalpy jump condition for A_{kf} is given by

$$\rho_k h_k(\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k - P_k(\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k + \underline{J}_{qk} \cdot \underline{n}_k$$

$$+ \rho_f h_f(\underline{U}_f - \underline{U}_{kf}) \cdot \underline{n}_f - P_f(\underline{U}_f - \underline{U}_{kf}) \cdot \underline{n}_f + \underline{J}_{qf} \cdot \underline{n}_f = 0. \qquad (A.30)$$

Accordingly,

$$v^{-1} \int_{A_{kf}} (\rho_k h_k - P_k)(\underline{U}_k - \underline{U}_{kf}) \cdot \underline{n}_k \, dA + v^{-1} \int_{A_{kf}} \underline{J}_{qk} \cdot \underline{n}_k \, dA$$

$$+ v^{-1} \int_{A_{kf}} (\rho_f h_f - P_f)(\underline{U}_f - \underline{U}_{kf}) \cdot \underline{n}_f \, dA$$

$$+ v^{-1} \int_{A_{kf}} \underline{J}_{qf} \cdot \underline{n}_f \, dA = 0. \qquad (A.31)$$

Summing up Eq. 3.58 for all k's and using Eq. A.30 yield

$$\gamma_v \frac{\partial}{\partial t} {}^{3i}\langle \rho_m h_m - P_m \rangle + \nabla \cdot \gamma_A {}^{2i}\langle (\rho_m h_m - P_m) \underline{U}_m \rangle$$

$$+ v^{-1} \sum_k \int_{A_{kf}} (\rho_k h_k - P_k)(\underline{U}_k - \underline{U}_s) \cdot \underline{n}_k \, dA$$

$$= - \gamma_v {}^{3i}\langle P_m \nabla \cdot \underline{U}_m \rangle - \nabla \cdot \gamma_A {}^{2i}\langle \underline{J}_{qm} \rangle$$

$$+ \gamma_v {}^{3i}\langle J_{Em} + \Phi_m + \dot{Q}_{wm} \rangle$$

$$- \nabla \cdot \gamma_A \sum_k \alpha_k {}^{2i}\langle (\rho_k h_k - P_k)(\underline{U}_k - \underline{U}_m) \rangle$$

$$- \gamma_v \sum_k \alpha_k {}^{3i}\langle P_k \nabla \cdot (\underline{U}_k - \underline{U}_m) \rangle, \qquad (A.32)$$

which, for constant ρ_k for <u>all</u> k's, becomes

$$\gamma_v \frac{\partial}{\partial t} {}^{3i}\langle \rho_m h_m \rangle + \nabla \cdot \gamma_A {}^{2i}\langle \rho_m h_m \underline{U}_m \rangle$$

$$+ v^{-1} \sum_k \int_{A_{kf}} \rho_k h_k (\underline{U}_k - \underline{U}_s) \cdot \underline{n}_k \, dA$$

$$= - \nabla \cdot \gamma_A {}^{2i}\langle \underline{J}_{qm} \rangle + \gamma_v {}^{3i}\langle J_{Em} + \Phi_m + \dot{Q}_{wm} \rangle$$

$$- \nabla \cdot \gamma_A \sum_k \alpha_k \, {}^{2i}\langle \rho_k h_k (\underline{U}_k - \underline{U}_m) \rangle. \tag{A.33}$$

4. Pure Stratified Flow

The TRAC code (Liles et al., 1979) equations can be modified for the present system to account for the porous media for the case of pure stratified flow, or where such an approximation is valid. When these equations are summed over all phase k in the fluid mixture m, we get, from Eqs. 4.5-4.7, or Eqs. A.12-A.14 and A.26:

$$\gamma_v (\partial \bar{\rho}_m / \partial t) + \nabla \cdot (\bar{\rho}_m \underline{U}_m) = 0, \tag{A.34}$$

$$\gamma_v (\partial \bar{\rho}_m \underline{U}_m / \partial t) + \nabla \cdot (\gamma_A \bar{\rho}_m \underline{U}_m \underline{U}_m) = \gamma_v \nabla P + \nabla \cdot \gamma_A \, \underline{\tau}_m + \underline{\tau}_m^T$$

$$+ \underline{f} \, \bar{\rho}_m - \sum_k \nabla \cdot \gamma_A \, [\alpha_k \rho_k (\underline{U}_k - \underline{U}_m)(\underline{U}_k - \underline{U}_m)], \tag{A.34}$$

$$\gamma_v (\partial \bar{\rho}_m u_m / \partial t) + \nabla \cdot (\gamma_A \bar{\rho}_m u_m \underline{U}_m) = - P \nabla \gamma_A \, \underline{U}_m$$

$$- \nabla \cdot \gamma_A \, (\underline{J}_{qm} + \underline{J}_{qm}^T) + \gamma_v J_E$$

$$- \nabla \cdot \sum_k \gamma_A \rho_k (\underline{U}_k - \underline{U}_m)(u_k - u_m), \tag{A.35}$$

where the superscript T refers to the turbulent contribution to the fluid stresses or heat fluxes. These equations are similar to those in Liles et al. (1979), except with the porosity and surface permeability accounted for.

5. Highly Dispersed Mixture

When applied to a highly dispersed mixture, Eq. A.14 becomes

$$\gamma_v (\partial \bar{\rho}_m \underline{U}_m / \partial t) + \nabla \cdot (\gamma_A \bar{\rho}_m \underline{U}_m \underline{U}_m) = \nabla \, P + \nabla \cdot \underline{\tau}_m + \underline{\tau}_m^T + \underline{f} \, \bar{\rho}_m \tag{A.36}$$

and Eq. 3.26, because of Eq. 4.8, gives

$$\gamma_v (\partial \alpha_k \bar{\rho}_k \underline{U}_k / \partial t) + \nabla \cdot (\gamma_A \alpha_k \bar{\rho}_k \underline{U}_k \underline{U}_k) = - \nabla \gamma_v \, \alpha_k \, P + \nabla \cdot \gamma_A \, \underline{\tau}_{mk} + \underline{\tau}_{mk}^T$$

$$+ \gamma_v \underline{V}_k + \gamma_v \Gamma_k \underline{U}_m + \gamma_v \, \underline{f} \alpha_k \bar{\rho}_k + \nabla \cdot \gamma_A \alpha_k \bar{\rho}_k (\underline{U}_k - \underline{U}_m)(\underline{U}_k - \underline{U}_m). \tag{A.37}$$

The general case of two-phase flow through a porous body calls for a description somewhere between this and the pure stratified flow.

ACKNOWLEDGMENT

We are indebted to Drs. R. T. Curtis, C. N. Kelber, and P. M. Wood of the United States Nuclear Regulatory Commission, for without their support this work would not have been possible. The painstaking care which Sally Moll and Brenda Wright exercised in the typing of this manuscript is also sincerely appreciated.

The work is performed under the auspices of the Reactor Safety Research Division of the U. S. Nuclear Regulatory Commission under Contract A-2045.

REFERENCES

Campbell, I. J., and A. S. Pitcher (1958), "Shock Waves in Liquid Containing Gas Bubbles," Proc. Roy. Soc. Ser. A243, 534.

Domanus, H. M., W. T. Sha, V. L. Shah, J. G. Bartzis, J. L. Krazinski, C. C. Miao, and R. C. Schmitt (1980) "COMMIX-2: A Steady/unsteady, Single-phase/ Two-phase Three-dimensional Computer Program for Thermal Hydraulic Analysis of Reactor Components," ANL-81-10, NUREG/CR-1807.

Grey, W. G. (1975), "A Derivation of the Equation for the Multiphase Transport," Chem. Eng. Sci. 30, 229-233.

Harlow, F. H., and A. A. Amsden (1975), "Numerical Calculation of Multiphase Flow," J. Comp. Physics, 17, 19-52.

Ishii, M., and K. Mishima (1980), "Study of Two-fluid Model and Interfacial Area," ANL-80-111, NUREG/CR-1873.

Kac, M. (1959) Probability and Related Topics in Physical Sciences, Interscience, New York.

Liles, D. R., et al. (1979), "TRAC-P1A, An Advanced Best-Estimate Computer Program for PWR/LOCA Analysis," Report LA-7777-MS, NUREG/CR-0665, Los Alamos Scientific Laboratory, Post Office Box 1663, Los Alamos, New Mexico 87545.

Sha, W. T. (1978), "A New Approach for Rod-Bundle Thermal-Hydraulic Analysis," Proc. of the International Meeting on Nuclear Power Reactor Safety, Brussels, Belgium, October 16-19, 1978.

Sha, W. T., et al. (1978), "COMMIX-1: A Three Dimensional Transient Single-Phase Component Computer Program for Thermal Hydraulic Analysis," NUREG/CR-0415, ANL-77-96.

Sha, W. T., and B. T. Chao (1981), "Local Volume-averaged Transport Equations for Single-Phase Flow in Regions Containing Fixed, Dispersed Heat Generating Solids," ANL-80-124, NUREG/CR-1969.

Sha, W. T., and J. C. Slattery (1980), "Local Volume-Time Averaged Equations of Motion of Dispersed and Turbulent Multiphase Flows," NUREG/CR-1491, ANL-80-51.

Sha, W. T., and S. L. Soo (1979), "On the Effect of P $\nabla\alpha$ Term in Multiphase Mechanics," Int. J. Multiphase Flow 5, 153-158.

Sha, W. T., and S. L. Soo (1980), "Addendum to Multiphase Fluid Mechanics," Int. J. Heat Mass Transfer 23, 583-586.

Sha, W. T., and S. L. Soo (1981), "Interface Transfer Integrals and Deviations in Volume-averaged Multiphase Flows," ANL report to be published.

Slattery, J. C. (1967), "Flow of Viscoelastic Fluids through Porous Media," AIChE J. 13, 1066.

Soo, S. L. (1967), "Fluid Dynamics of Multiphase Systems," Blaisdell.

Soo, S. L. (1968), "Nonequilibrium Fluid Dynamics - Laminar Flow over a Flat Plate," ZAMP 19, 543-563.

Soo, S. L. (1978), "Diffusion in Multiphase Flow," Iran J. Sci. Technol., 7, 31-35.

Soo, S. L. (1980), "Equations of Multiphase-Multidomain Mechanics," Multiphase Transport (T. Veziroglu, Ed.) Hemisphere Publ. Co., Washington, D.C., 1, 291-305.

Soo, S. L. (1981), "Effects of Configuration of Phases on Dynamic Relations," AIChE Symp. Series, 208, 77 Heat Transfer (R. P. Stein, Ed.) 152-160.

Soo, S. L., and W. T. Sha (1975), "Porous Medium Formulation of Three-Dimensional Transient Flow through Rod Bundles," Turbulent Forced Convection in Channels and Rod Bundles (S. Kakac and D. B. Spalding, eds.) Hemisphere Publ. Co., Washington, D.C., Vol. 1, 301-323.

Whitaker, S. (1967), "Diffusion and Dispersion in Porous Media," AIChE J. 13, 420-427.

Whitaker, S. (1969), "Advances in Theory of Fluid Motion in Porous Media," Ind. Eng. Chem. 61(12), 14.

Distribution for NUREG/CR-2354 (ANL-81-69)

Internal:

E. S. Beckjord
C. E. Till
R. S. Zeno
W. L. Baumann
A. R. Brunsvold
B. C. Chen
T. H. Chien
H. M. Domanus
D. M. France
P. Kehler
J. J. Lorenz
R. W. Lyczkowski
C. C. Miao
W. T. Sha (5)
V. L. Shah
J. E. Sullivan
S. P. Vanka
W. L. Chen
D. H. Cho
M. Ishii
D. R. Ferguson
D. P. Weber
E. M. Gelbard
L. W. Deitrich
R. M. Singer
Y. W. Shin
G. K. Leaf
P. B. Abramson
H. H. Hummel
P. L. Garner
ANL Patent Dept.
ANL Contract File
ANL Libraries (2)
TIS Files (3)

External:

USNRC, Washington, for distribution per R7 (295)
DOE-TIC (2)
Manager, Chicago Operations Office, DOE
President, Argonne Universities Association, Argonne, Illinois
Components Technology Division Review Committee:
- A. A. Bishop, U. Pittsburgh, Pittsburgh, Pa. 15261
- F. W. Buckman, Consumers Power Co., 1945 Parnall Rd., Jackson Mich. 49201
- W. M. Jacobi, Nuclear Fuel Div., Westinghouse Electric Corp., Pittsburgh, Pa. 15230
- M. A. Schultz, North Palm Beach, Fla. 33408
- E. E. Ungar, Bolt, Beranek and Newman, Inc., Cambridge, Mass. 02138
- J. Weisman, U. Cincinnati, Cincinnati, Ohio 45221
- R. A. Greenkorn, Purdue U., West Lafayette, Indiana 47907

B. T. Chao, U. Illinois, Urbana, Ill. 61801
M. S. Kazimi, Massachusetts Inst. Tech., Cambridge, Mass. 02139
G. S. Lellouche, Electric Power Research Inst., Palo Alto, Calif. 94303
R. Markley, Westinghouse Advanced Reactors Div., Madison, Pa. 15663
B. R. Sehgal, Electric Power Research Inst., Palo Alto, Calif. 94303
J. C. Slattery, Northwestern U., Evanston, Ill. 60201
S. L. Soo, U. Illinois, Urbana, Ill. 61801
J. F. Thompson, Mississippi State U., Mississippi State, Miss. 39762
N. E. Todreas, Massachusetts Inst. Technology, Cambridge, Mass. 02139